Split States, Entropy Enigmas, Holes and Halos

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Split States, Entropy Enigmas, Holes and Halos

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Abstract: We investigate degeneracies of BPS states of D-branes on compact Calabi-Yau manifolds. We develop a factorization formula for BPS indices using attractor flow trees associated to multicentered black hole bound states. This enables us to study background dependence of the BPS spectrum, to compute explicitly exact indices of various nontrivial D-brane systems, and to clarify the subtle relation of Donaldson-Thomas invariants to BPS indices of stable D6-D2-D0 states, realized in supergravity as "hole halos." We introduce a convergent generating function for D4 indices in the large CY volume limit, and prove it can be written as a modular average of its polar part, generalizing the fareytail expansion of the elliptic genus. We show polar states are "split" D6-anti-D6 bound states, and that the partition function factorizes accordingly, leading to a refined version of the OSV conjecture. This differs from the original conjecture in several aspects. In particular we obtain a nontrivial measure factor $g_{\rm top}^{-2}\, e^{-K}$ and find factorization requires a cutoff. We show that the main factor determining the cutoff and therefore the error is the existence of "swing states" — D6 states which exist at large radius but do not form stable D6-anti-D6 bound states. We point out a likely breakdown of the OSV conjecture at small $g_{\rm top}$ (in the large background CY volume limit), due to the surprising phenomenon that for sufficiently large background Kähler moduli, a charge $\Lambda\,\Gamma$ supporting single centered black holes of entropy $\sim \Lambda^2 S(\Gamma)$ also admits two-centered BPS black hole realizations whose entropy grows like Λ^3 when $\Lambda \to \infty$.

Contents

1. Introduction

String theory has been spectacularly successful in microscopically reproducing the entropy of certain classes of black holes, in particular of supersymmetric charged black holes. What made this possible is the fact that for these black holes, the entropy can be identified with the logarithm of the Witten index of the system, which is independent of the string coupling constant, enabling one to count states in the zero coupling limit where the D-brane description becomes accurate [1]. Alternatively one can use the M-theory description, as was done in [2] for four dimensional D4-D2-D0 BPS black holes in IIA Calabi-Yau compactifications. In this case the relevant weak coupling limit is the limit of large Calabi-Yau volume and large M-theory circle radius.

However, until recently all these derivations were limited to cases dual to systems in regimes in which some form of the Cardy formula could be applied. For example the computation of [2] was limited to zero D6-brane charge and large D0-charge N. The restriction to large N followed from the use of the Cardy formula. The parallel derivation in the D4-D0 picture in string theory [3] is restricted to the same large D0-charge regime. Thus the standard treatments are in fact valid only in a very small subset of charge space. In particular, in this regime none of the IIA worldsheet instanton contributions to the supergravity entropy are visible.

Further significant progress on the microscopic accounting of entropy only came after the supergravity prediction for the entropy, based on the attractor mechanism [4, 5], was refined in [6, 7, 8, 9], leading in turn to the formulation of a famous conjecture by Ooguri, Strominger and Vafa (OSV) [10]. The OSV conjecture predicts a far-reaching generalization of the correspondence between supergravity and statistical entropies, refining it to all orders in a $1/Q$ expansion, Q being some measure of the charge. One way of stating the original conjecture is

$$\Omega(p, q) \sim \int d\phi \, e^{-2\pi\phi^\Lambda q_\Lambda} \, |\mathcal{Z}_{\text{top}}(g_{\text{top}}, t)|^2, \tag{1.1}$$

where $\Omega(p, q)$ is a suitable index of BPS states of given charge $\Gamma = (p, q)$, defined below in (1.6), and $\mathcal{Z}_{\text{top}}(g_{\text{top}}, t)$ is the topological string partition function with certain (p, ϕ) dependent substitutions for the topological string coupling g_{top} and Kähler moduli t, also detailed below in section 1.3. By construction, the leading saddle point approximation to (1.1) is $e^{S_{\text{BHW}}(p,q)}$, where S_{BHW} is the Bekenstein-Hawking-Wald entropy obtained from the standard $\mathcal{N} = 2$ low energy two derivative action plus F-term R^2 corrections, governed by topological string amplitudes.

A version of the conjecture counting BPS states on *noncompact* Calabi-Yau manifolds was subsequently investigated in many examples. The reason for considering noncompact Calabi-Yau manifolds is that much more is known about the D-brane systems they carry and the counting of their BPS states. On the other hand the immediate black hole interpretation is lost, and one should be wary of drawing conclusions for the compact case from the noncompact case. In this paper, we will strictly limit ourselves to the compact case. Direct tests for this case have been more limited [11, 12, 13], mainly because little is known about the behavior of various curve counting invariants at large degree in this compact setting. However, more recently progress was made towards model independent derivations of the conjecture [16, 17, 18, 19]. As a byproduct, these studies have opened the window to extensions of microscopic derivations of black hole entropy beyond the "Cardy regime," and in particular to give an explanation for the appearance of IIA worldsheet instanton corrections to the entropy.

Nevertheless, the situation is still far from being completely understood, and several problems were left open in these recent studies, some explicitly, some only implicitly.

One of the problems in the first category is the fact that there is still no general derivation for the case with nonzero D6-brane charge. Another one is the need to keep the black hole attractor point within some sufficiently small neighborhood of the infinite radius limit, requiring in particular the magnetic D4 charge P to lie within the Kähler cone (excluding in particular the so-called "small" black holes). These limitations also hold for the present work.

The more subtle problems on the other hand are related to the intrinsic ambiguities present in (1.1), some of which were already pointed out in [10, 11, 12]. The most serious ones are:

- The indices $\Omega(p, q)$ in fact depend on the boundary conditions of the scalar fields at infinity. Denoting the background by t_∞ we should, and henceforth will, denote the indices by $\Omega(p, q; t_\infty)$. The t_∞ dependence is due to jumps at walls of marginal stability. On the other hand the right hand side of (1.1) does not have this dependence. This raises the question: For which value of t_∞ is the conjecture is supposed to hold?

- Since $\mathcal{Z}_{\rm top}$ is divergent and only makes sense as an asymptotic perturbative expansion, it is clear that the conjecture can at most hold approximately. However, it is not clear *a priori* what the regime of validity should be, nor what the order of the error is, nor even how to define properly the integral (1.1). In other words, it is not clear what "$\sim$" means.

- It is not clear whether there should be an additional integration measure factor in (1.1).

The first issue has been sidestepped in most studies of the OSV conjecture so far. Upon closer inspection though, one sees that typically an implicit choice of t_∞ is made. For example if one counts states in a classical geometric brane picture, one is implicitly working in the infinite radius limit, i.e. $t_\infty = i\infty$. This will also be the value of t_∞ considered in this paper. We should stress that this is different from working with local Calabi-Yau manifolds. First, we are taking a well defined limit $t_\infty \to i\infty$ of the full, compact degeneracies, without making truncations of degrees of freedom as one does in local models. Moreover, by simultaneously tuning the IIA string coupling one could actually keep the M-theory CY volume $V_M \sim V_{\rm IIA}/g_{\rm IIA}^2$ finite in this limit.[1]

The second issue has been largely ignored by keeping derivations formal and not worrying about issues of convergence or how to define the right hand side of (1.1) such that it makes sense as an integral. In [18] the need for a cutoff in $\mathcal{Z}_{\rm top}$ and the existence of corrections to the OSV formula were emphasized but the analysis was not sufficiently detailed to provide a precise description of either one of those. For certain $\mathcal{N} \geq 4$ models [11, 12, 13, 14, 15], where cutoffs are not needed due to the simplicity of the topological string partition function, explicit exponential corrections were found.

Finally, the need for an additional measure factor was was pointed out in [12] for small black holes, in [13] for T^6 and $T^2 \times K3$ compactifications, in [15] for $\mathcal{N} = 4$ models, and on general theoretical grounds in [20]. On the other hand, none of the derivations [17, 18] detected an additional measure factor.

Our analysis will tackle all these problems head-on for arbitrary compact (proper) Calabi-Yau manifolds, resulting in a refined, unambiguous version of the conjecture at $t_\infty = i\infty$, with a precise cutoff prescription for $\mathcal{Z}_{\rm top}$. Moreover, we will in fact find a nontrivial extra measure factor

$$\mu \sim g_{\rm top}^{-2}\, e^{-K} \tag{1.2}$$

in agreement with previous special case studies [11, 12, 13, 14]. The presence of a similar nontrivial measure factor has been previously discussed from a different point of view in [15]. Finally we will formulate detailed constraints on (p, q) for the conjecture to hold to exponential accuracy, and give a concrete error estimate within this domain of validity somewhat larger than that found in previous special case studies [13].

A more or less self-contained technical summary of our final result — the refined OSV formula — can be found in section 7.

The main challenge, to which much of the paper is devoted, is essentially controlling the error and ensuring it does not swamp the effects of interest. We outline the issues involved a bit further on in section 1.2, and discuss a number of unresolved problems in this context.

1.1 Outline

Besides a precise version of the OSV conjecture as outlined above, we will obtain several results of independent interest:

1. In section 2 we show that the "black hole partition function" of OSV for $p^0 = 0$ may be obtained from a well-defined, convergent (topologically twisted) D4 partition function $\mathcal{Z}_{D4}$ through a formal substitution of arguments. We then demonstrate, using TST duality, that $\mathcal{Z}_{D4}$ transforms as a generalized multi-variable Jacobi form under modular transformations. From this, we rigorously establish a "fareytail expansion"

[1] This indicates that M-theory is perhaps the most natural framework to consider this limit. We will develop most of our picture in IIA, but the fundamental building blocks we will use, namely attractor flow trees, are universal and can equally well be interpreted in IIA, IIB or M-theory, or even microscopically, as we will see.

[26, 27] for the fareytail-transform $\widehat{\mathcal{Z}}_{D4}$, as well as one for the original $\mathcal{Z}_{D4}$. This expresses the D4D2D0 indices of arbitrary charges in terms of those of a distinguished, finite set of "polar" charges, which have reduced D0-charge $\hat{q}_0 > 0$. The use of the fareytail expansion in this problem was suggested in [11] and is dual to the M-theory derivations of [18, 28, 47]. The result obtained here refines and extends these results. This section is logically quite independent of the remainder of the paper, although the final result (2.71) will be used in the derivation in section 6.4.2.

2. In section 3, we give a review of the four dimensional supergravity picture of BPS bound states as multicentered black hole "molecules" and of the phenomenon of decay at marginal stability in this setting. We emphasize the power of attractor flow *trees* in establishing the existence of such solutions, and argue in general that these flow trees give a useful partition of the classical BPS configuration moduli space and quantum BPS Hilbert space. We give very concrete explicit examples of D6-anti-D6 two centered bound states, halos, Sun-Earth-Moon systems, and iterations of those.

   Within the class of two-centered black hole examples, we encounter a rather surprising phenomenon: when one uniformly scales up a generic ($P > 0$) D4-D2-D0 charge Γ as $\Gamma \to \Lambda\Gamma$, one finds that for Λ sufficiently large and in a background with $\operatorname{Im} t_\infty$ sufficiently large, there always exist two-centered black hole BPS bound states whose horizon entropy is parametrically larger than the single centered horizon entropy. More precisely the horizon entropy of these two centered solutions grows as Λ^3, while the single centered entropy only grows as Λ^2. Although this is easily seen to be fully compatible with holography (as all distances scale as $\Lambda^{3/2}$), it is still quite unexpected, and appears at odds with the OSV formula (at $t_\infty \to i\infty$) in this limit, as that formula predicts $\log|\Omega| \sim \Lambda^2$. We refer to this phenomenon as the "entropy enigma".

   We also demonstrate the existence of an interesting class of multiparticle "scaling" solutions, which are characterized by a configuration scale modulus λ such that in the limit $\lambda \to 0$, the solution becomes indistinguishable from a single centered black hole to a distant observer, while a near observer keeps on seeing nontrivial microstructure.

   Finally, we show that the polar states forming the basis of the fareytail expansion correspond to charges which do not have a single centered black hole description. Instead they are realized as BPS black hole configurations consisting of two (or more) clusters of nonzero opposite D6 charges. In this sense polar states are "split states". This split nature will translate into approximately factorized degeneracies, which we show in later sections to give rise eventually to the factorization $\mathcal{Z}_{\rm BH} \sim \mathcal{Z}_{\rm top}\overline{\mathcal{Z}_{\rm top}}$, i.e. the OSV conjecture.

3. In section 4 we briefly review the microscopic counterparts of these multicentered configurations in terms of stretched open strings and tachyon condensation. We also exhibit how to a certain extent the split nature of polar states is mirrored even in the large radius geometrical description of these D-brane states, by matching charges and moduli spaces.

4. In section 5, we get to the actual counting of BPS states and describe perhaps the most important result in the paper. We give physical arguments for a wall crossing formula giving the jump $\Delta\Omega(\Gamma;t)$ of the index at a wall of marginal stability $t = t_{\rm ms}$ corresponding to a decay $\Gamma \to \Gamma_1 + \Gamma_2$. The index changes by

$$\Delta\Omega(\Gamma;t) = (-1)^{\langle\Gamma_1,\Gamma_2\rangle-1}|\langle\Gamma_1,\Gamma_2\rangle|\;\Omega(\Gamma_1;t_{\rm ms})\;\Omega(\Gamma_2;t_{\rm ms}) \tag{1.3}$$

when Γ_1 and Γ_2 are primitive. Of course, for fixed Γ_1, Γ_2 this is also the wall of marginal stability for other charges $\Gamma_{N_1,N_2} \to N_1\Gamma_1 + N_2\Gamma_2$, $N_1, N_2 > 0$. We show that the formula for $N_1 = 1$ but arbtirary N_2 is most conveniently given in terms of a generating function:

$$\sum_{N_2>0} \Delta\Omega(\Gamma_1 + N_2\Gamma_2)\, q^N = \Omega(\Gamma_1) \prod_{k>0} \Big(1 - (-1)^{k\langle\Gamma_1,\Gamma_2\rangle}\, q^k\Big)^{k|\langle\Gamma_1,\Gamma_2\rangle|\,\Omega(k\Gamma_2)} \tag{1.4}$$

where all indices are understood to be evaluated at $t = t_{\rm ms}$. These formulae in turn give rise to a powerful factorization formula for BPS indices based on attractor flow trees.

We verify these formulas explicitly for a number of nontrivial bound states of branes described by quivers and/or large radius sheaves, including a three node quiver with a closed loop and a generic cubic superpotential. For this case we find an intriguing exact formula for the indices in terms of an integral of the product of three Laguerre functions. This shows a phase transition (as a function of the charges) in the growth of the degeneracies, going from polynomial to exponential exactly at the transition point where the black hole-like scaling solutions mentioned above come into existence. Moreover in this regime we find the rather suggestive asymptotics $\Omega \sim 2^{I_{12}}\, 2^{I_{23}}\, 2^{I_{31}}$, where I_{ij} denotes the number of arrows between the respective nodes in the quiver (note that these grow quadratically with uniform charge scalings, making this a macroscopic entropy).

5. In section 6 we turn to the counting of the BPS states specifically relevant for our derivation of the OSV conjecture.

   In section 6.1, we analyze the spectrum of D6-D2-D0 BPS bound states with unit D6 charge and the generating function $\mathcal{Z}_{\rm D6-D2-D0}|_{t_\infty}$ for their indices in a given background $t_\infty = B + iJ$. We discuss the relation of these generating functions with the Donaldson-Thomas(DT) / Gopakumar-Vafa (GV) partition functions. It turns out that for D6D2D0 states realized in supergravity as D2D0 "halos" around a core with nonzero D6 charge, there are walls of marginal stability which run all the way out to infinite Kähler class, leading to jumps in $\mathcal{Z}_{\rm D6-D2-D0}$ when the B-field is varied, and hence to explicit deviations of physical stability from μ-stability even in the infinite CY volume limit. Thus one can only potentially identify $\mathcal{Z}_{\rm D6-D2-D0}$ with the DT partition function in certain limits of the background. This includes particular limits $B \to \infty$, for which we show that the contribution of all stable halo states to $\mathcal{Z}_{\rm D6-D2-D0}$ is given exactly by the genus $r = 0$ factor of the GV/DT infinite product. We argue that in such limits we can indeed identify $\mathcal{Z}_{DT} = \mathcal{Z}_{\mathrm{D6-D2-D0}}$, refining the arguments and result of [29]. Under this identification, the genus $r > 0$ part of the DT infinite product counts "core" states, which are stable for any value of the B-field at infinite Kähler class.

Sections 6.2 and 6.3 contain the most subtle steps in our derivation of the OSV conjecture. We will outline the origin of the complications and discuss the unresolved issues that arose separately in section 1.2 below. In section 6.2 we analyze the D6-anti-D6 type bound states giving rise to polar D4-D2-D0 states, and address in particular the question of which D6 and anti-D6 states correspond to "extreme" polar states, i.e. states whose reduced D0-charge $\hat{q}_0$ is near-extremal. Restriction to these states in the generating function is necessary to obtain exact factorization, which we further discuss in section 6.3. Finally, in section 6.4, by combining the results of the previous sections, this leads to to a derivation of a refined version of the OSV formula.

6. In section 7, we give a thorough discussion of our final result.

7. The seven appendices contain numerous technical details and several additional results. In appendix A, we collect some definitions and conventions, and in appendix B we summarize a number of results in algebraic geometry we use. In appendix C, we prove a partial result regarding the finiteness of the number of split attractor flows. In appendix D we outline an efficient algorithm for checking numerically the existence of attractor flow trees. These algorithms helped in checking the extreme polar state conjecture. In appendix E we give the details of the computation of the closed loop three node quiver index mentioned above. In appendix F, we clarify some confusion which existed in the literature regarding whether one should compare the index or total degeneracy computed at zero string coupling to the black hole entropy (in particular in five dimensions, where it seemed that the former gave wrong results), and show it is in fact the index, if one uses the proper one. Finally, in G, we give an independent derivation of our version of the OSV formula in the $g_{\mathrm{top}} \to \infty$ regime, using techniques originally developed in [30, 31] to count closed string flux vacua.

1.2 Challenges for a complete proof and unresolved issues

Although the basic idea underlying our derivation of the OSV conjecture is quite simple, turning it into a complete proof proved to be a rather complex task, and we have only been partially successful. This is not a shortcoming of the IIA picture we work in — the same would be true if one wanted to turn the ideas of [17, 18] into an actual proof, and the complications outlined below have all direct equivalents in the M-theory picture used there. We elaborate on this in section 7.3.

At the core of the complexity lies the fact that in order to obtain the factorized form of the integrand in (1.1) it is necessary to introduce a cutoff. The factorization ultimately comes from the fact that, through the fareytail series of section 2, all D4 indices can be expressed in terms of the indices of a finite number of polar D4 states, which as we mentioned above do not form single centered black holes but can be described as D6-anti-

D6 bound states.[2] In a suitable limit of the background, single D6-states are counted by DT invariants, which determine $\mathcal{Z}_{\rm top}$. If there were a one-to-one map between all polar D4 states and all possible pairs of single D6 and single anti-D6 states in the background in which they are counted by DT invariants, we would thus get exact factorization and a strong version of the OSV conjecture.[3]

Unfortunately, this is not the case. Not all polar states are *single* D6 - anti-D6 bound states, and moreover the subset of single D6-anti-D6 pairs giving rise to actual bound states is rather limited and complicated. In particular, even within the set of pairs which do form bound states, the two elements of the pair cannot in general be chosen independently, ruining exact factorization.

However, we will argue that for sufficiently polar states, i.e. D4 states with charges sufficiently close to those of a pure D4, the desired one-to-one correspondence does indeed hold. [4] "Sufficiently close" is measured by a parameter $\eta \geq 0$, defined in section 6.2.2, with $\eta = 0$ corresponding to the pure D4, which can be realized as a bound state of a pure D6 and a pure anti-D6 with suitable fluxes turned on on their worldvolumes. More precisely $\eta := \frac{\hat{q}_0 - (\hat{q}_0)_{\rm max}}{(\hat{q}_0)_{\rm max}}$.

Dropping all polar states with $\eta > \eta_*$ with η_* sufficiently small allows us to obtain an approximate factorized formula. The error introduced in this way turns out to amount to a multiplicative correction to the integrand in (1.1) of order

$$\exp\left[\mathcal{O}(e^{-\eta_* g_{\rm top} P^3})\right]. \tag{1.5}$$

Now the essence of the OSV conjecture is that D-brane BPS degeneracies are expressed in terms of the data of worldsheet instanton effects. These effects contribute factors $\exp\left[\mathcal{O}(e^{-g_{\rm top} P})\right]$ to the integrand, where we used the relation ${\rm Im}\, t \sim g_{\rm top} P$ detailed below in (1.10). Thus, to keep the error smaller than the effects of interest, we need to keep $\eta_* \gg 1/P^2$.

Much of our work is aimed at investigating how large one can take η_* without ruining factorization. This requires getting sufficient control on the very difficult problem of stability of BPS bound states. For this purpose, we spend some effort extending the development of the theory of attractor flow trees [22, 21, 23, 24]. This framework gives in principle a way to study systematically stability issues. We successfully used it to get large parts of the problem under control, but regrettably a few gaps remain.

We make these gaps explicit by formulating a number of precise conjectures. The first one is the "split attractor flow conjecture," formulated in section 3.2.2. This states essentially that we can classify BPS states by attractor flow trees, and that the number of such trees of a given total charge is finite. There are very good physical arguments for

[2] Here and in the following, by D4 and D6 states, we mean states with arbitrary induced lower dimensional charges.

[3] Actually there would still be a series of fareytail corrections, but these are under exact control, and each of these corrections would again be factorized.

[4] By a "pure D4" we mean a D4-brane BPS state with $N = 0, F = 0$ in the notation of section 2.1 below. It has charge $\Gamma = P + (P^3 + c_2 \cdot P)/24$. Similarly, by a "pure D6" we mean a rank 1 D6 brane with no lower charges, i.e. $\Gamma = 1$. A "pure fluxed D6" means $\Gamma = e^S$.

this, and the analysis of this paper gives ample evidence for it. We have no reasons to doubt it. The second one is the "extreme polar state conjecture," formulated in section 6.2.2. This states that all polar states with $\eta < \eta_*$ sufficiently small can be realized as single D6-anti-D6 bound states with the charges of the constituents being close to those of the pure fluxed D6 and anti-D6 branes describing the pure D4. "Close" in this case is measured by a parameter ϵ defined in section 6.2, eq. (6.51), with $\epsilon \sim \eta_*$. We give some physical arguments and considerable numerical and analytical evidence for the extreme polar state conjecture, and we strongly believe it to be true.

Modulo one assumption, these conjectures then allow us to show that we get the required factorization for a sufficiently small but P-independent value of η_*, in which case indeed $\eta_* \gg 1/P^2$ in the large P limit, making the error exponentially smaller than the instanton contributions. That assumption is that the BPS indices of the D6 and anti-D6 charges restricted by the cutoff ϵ do not jump between the region in moduli space where they equal DT invariants and the region in moduli space where the central charges of the two constituents line up.

Unfortunately, this assumption turns out to be *wrong* at large P with fixed ϵ ! There can be D6 or anti-D6 states within the ϵ bound which *do* decay between these loci in moduli space. Such pairs cannot combine into D6-anti-D6 BPS bound states, again spoiling the desired factorization, and moreover spoiling the identification of the D6 indices with DT invariants and the topological string.

Such states, which we call "swing states," in fact do exist, at least when $\eta_* \sim \epsilon > \mathcal{O}(1/P)$, as we discuss in section 6.3.2. When $\epsilon < \mathcal{O}(1/P^3)$, we can prove in general that swing states are absent at sufficiently large P. Let ξ_{cd} be the minimal value of ξ such that taking $\epsilon < \delta/P^{\xi}$, swing states are absent at sufficiently large P for some fixed δ (for reasons which will become clear in section 6.3.2, we call this the "core dump exponent"). From what we just said, we know that $1 \leq \xi_{cd} \leq 3$. But given the relation $\eta_* \sim \epsilon$, we need $\xi_{cd} \leq 2$ for the error not to be parametrically larger than the instanton contributions. We suspect that in fact $\xi_{cd} = 1$, and give some circumstantial evidence for this claim, but are not fully confident, so we consider this to be an unresolved issue. Note that in this case, the corrections are of order $e^{-g_{\rm top} P^2} \sim e^{-(\operatorname{Im} t)^2/g_{\rm top}}$, suggestive of D4/M5 corrections to the topological free energy. Indeed, the D6D2D0 swing states we find are realized in supergravity as two-centered D6-D4 bound states, which lift to M5 rings circling the center of Taub-NUT in M-theory.

Assuming $\xi_{cd} \leq 2$, there is no further obstacle to proving our refined OSV formula for $t_\infty = i\infty$, at least for $g_{\rm top} > \mathcal{O}(1)$, that is, at *strong* topological string coupling. The restriction to strong coupling might seem odd at first, as this is opposite to the regime for which the OSV conjecture was intended to be valid, but it becomes less so when one realizes that all the old successes of microscopically reproducing the Bekenstein-Hawking entropy, such as [2], in fact have saddle point values $g_{\rm top} \gg 1$, this being essentially equivalent to being in the regime of applicability of the Cardy formula. Also, all of the other recent studies based on reduction to a weakly coupled brane system [17, 18], although going beyond the $g_{\rm top} \gg 1$ limit, are upon closer inspection implicitly equally limited to the strong topological coupling regime $g_{\rm top} > \mathcal{O}(1)$. Moreover, the checks for small black holes

[11, 12] were in general only valid in the region of strong topological string coupling.

Technically, the reason for the restriction to $g_{\rm top} > \mathcal{O}(1)$ comes from the fact that the error estimate (1.5), which arises from dropping all polar states with $\eta > \eta_*$, is actually only manifestly valid for $g_{\rm top}$ sufficiently large. In this case, the most important contributions to the error come from polar states with η near the cutoff, as the ones with large values of η are exponentially suppressed. However, when $g_{\rm top}$ becomes smaller, the exponential suppression becomes weaker and at a certain point, the bulk of the polar terms (order 1 values of η) will start to dominate the original partition function and therefore the error produced by dropping them, because they have more entropy than the extreme polar ones. So something like a phase transition occurs, with $g_{\rm top}$ playing the role of inverse temperature. Simple estimates suggest that the degeneracies of the polar states at fixed η grow with P as $e^{\eta P^3}$, and the transition occurs at the value of $g_{\rm top}$ where this starts to dominate over the suppression factor $e^{-\eta\, g_{\rm top} P^3}$, hence at some $\mathcal{O}(1)$ value of $g_{\rm top}$. If the growth estimate is correct, then for values of $g_{\rm top}$ less than this, factorization breaks down.

This is not a failure of the derivation itself, but is in fact very closely related to the entropy enigma mentioned earlier — indeed, the supergravity configurations dominating the entropies of the polar states are precisely of the same kind as those giving rise to the entropy enigma. Moreover, since the saddle point value of $g_{\rm top}$ in (1.1) scales as $1/\Lambda$ when $(p,q) \to \Lambda(p,q)$, the large Λ regime is equivalent to the small $g_{\rm top}$ regime, and thus the appearance of the enigmatic configurations with entropy scaling as Λ^3 is consistent with the potential failure of the conjecture at weak $g_{\rm top}$ (and $t_\infty = i\infty$), which predicts $\log|\Omega| \sim \Lambda^2$

We can only say there is a "potential failure" here because there is a possible loophole which might still save the conjecture even in this large Λ regime. This loophole is discussed in detail in section 7. It is based on the fact that since $\Omega(p,q;t_\infty)$ is an *index*, it receives contributions of different signs from many multicentered black hole configurations, so there might in principle be miraculous cancelations altering the exponential growth from $e^{c\Lambda^3}$ down to $e^{c\Lambda^2}$. We argue this is very unlikely, but might still have a (remote) chance of being true if a similar cancelation happens for the DT invariants approximately building up our polar indices. The problem for DT invariants can be phrased in a mathematically precise way. The upshot is that if $N_{DT}(\beta,n)$ is a DT invariant for curve class β with $D0$ charge n then we should study the large λ asymptotics of $\log N_{DT}(\lambda^2\beta,\lambda^3 n) \sim \lambda^k$. (See eq. (7.26) for a more precise version.) The straightforward estimate based on the entropy of the corresponding D6D2D0 black holes suggests $k=3$, which would invalidate weak coupling OSV at $t_\infty = i\infty$. If, due to miraculous cancelations between different contributions to the DT invariants, we get $k \leq 2$, this would be suggestive of cancelations between the related actual indices, and perhaps the extension to weak coupling might still be possible (but this is by no means guaranteed). Although such cancelations might seem like ludicrous wishful thinking, we discuss a number of heuristic arguments pro (but also contra) this hypothesis.

Of course, it might also be that one should not take $t_\infty = i\infty$. Other natural prescriptions might be to take t_∞ to be at the attractor point $t_*(p,q)$ in (1.1) or to take t finite and fixed while sending $P \to \infty$. Both of these turn out automatically to eliminate the enigmatic Λ^3 configurations when $\Lambda \to \infty$, but would also spoil some of the interesting interpretations of the conjecture as an example of large radius D-brane gauge theory - gravity duality. Moreover, they would also push direct microscopic verification into a quantum geometric regime (due to the importance of α' correction at finite values of $\mathrm{Im}\, t_\infty$) which so far has proven intractable. We again refer to section 7 for more details.

Note added in version 2:

1. After version 1 of this paper appeared on the arXiv, the paper [25] appeared, in which the growth of DT invariants $\log N_{DT}(\lambda^2\beta, \lambda^3 n) \sim \lambda^k$ was numerically studied, based on available data sets of DT invariants of a number of compact CY manifolds. Although these data sets are too limited to directly extract asymptotics, one can get predictions for k by using Richardson transforms. Surprisingly, the results suggest $k = 2$, exactly the critical value for the OSV conjecture at $t = i\infty$ to have a chance of being correct even at weak topological string coupling, and implying the "miraculous cancelations" do indeed occur! Although as mentioned above and discussed at length in section 7.4, such cancelations at the level of DT invariants are not quite enough to make OSV work at weak coupling (for this one also needs cancelations at a more detailed level of different contributions to the D6D4D2D0 indices), it is clear that if the numerical results of [25] indeed correctly capture the $\lambda \to \infty$ asymptotics of the DT invariants, the unknown mechanism underlying these cancelations might conceivably also imply the more general cancelations required for weak coupling OSV. It would be extremely interesting to settle this issue.

2. We would like to stress that the implications of such miraculous cancelations could be enormous, going well beyond the issue of the range of validity of the OSV conjecture. In particular, if $k = 2$, then this raises the possibility that the Donaldson-Thomas partition function (1.20) gives a convergent, nonperturbative completion of the topological string partition function, of which (1.17) is a divergent asymptotic expansion. Indeed if $k = 2$ the sum over β will have a nonzero radius of convergence for fixed n. However proper convergence would actually also require $N_{DT}(\lambda^2\beta, \lambda^3 n)$ to vanish identically at sufficiently large λ for all strictly negative n, which requires an even more miraculous, exact cancelation to occur. Further work is needed to determine whether this might be the case or not. It is perhaps also worth noting that our equations (6.101)-(6.102) below in fact can be interpreted as defining a nonperturbative completion of the norm squared $|\mathcal{Z}_{\rm top}|^2$ of the topological string wave function, starting from the *polar part* $\mathcal{Z}^-$ of the D4 partition function, since when $P \to \infty$, $\mathcal{Z}^\epsilon_{DT}$ becomes $\mathcal{Z}_{DT}$.

1.3 Preliminaries

Let us conclude this section by reviewing some basic definitions which will be used in the text.

We will be studying type IIA D-branes wrapping cycles in a nonsingular compact Calabi-Yau 3-fold X of generic holonomy. The Hilbert space of type IIA on $\mathbb{R}^{1,3} \times X$ is graded by RR charge $\Gamma \in K^0(X)$. We ignore possible torsion subgroups and identify the charge group with $H^{\rm even}(X;\mathbb{Z})$, modulo torsion. Near a large radius limit of X there is a canonical electromagnetic decomposition $\Gamma = (p,q)$ with magnetic charges $p \in H^0(X;\mathbb{R}) \oplus H^2(X;\mathbb{R})$ and electric charges $q \in H^4(X;\mathbb{R}) \oplus H^6(X;\mathbb{R})$. Picking a basis $\{D_A\}_A$, $A = 1,\ldots,h := h^{1,1}(X)$ of $H^2(X,\mathbb{Z})$, the charges can be written in components as $p =: p^0 + P^A D_A$, $Q_A := \int_X D_A \wedge q$, $q_0 := \int_X q$. We also introduce an index Λ running over $0,1,\ldots,h$, and denote components of p by p^Λ and q by q_Λ. [5] The crucial boundary conditions on the fields at spatial infinity are those for the vectormultiplet scalar fields of the effective $\mathcal{N}=2$ supergravity defined on $\mathbb{R}^{1,3}$. For IIA compactifications these are the complexified Kähler moduli $t := t^A D_A := B + iJ$. In the superselection sector (p,q) there is a central charge $Z(p,q;t)$ of the $\mathcal{N}=2$ supersymmetry algebra and there is is a well-defined finite-dimensional space of BPS states $\mathcal{H}(p,q;t)$. These are the states at rest transforming in the small representations of the little $\mathcal{N}=2$ superalgebra. Alternatively, they are the 1-particle states satisfying the energy bound $E = |Z(p,q;t)|$.

As pointed out in [11] the appropriate index to use in this context is the second helicity supertrace. We will denote it as:[6]

$$\Omega(p,q;t) := -2\,\mathrm{Tr}_{\mathcal{H}(p,q;t)}\,(-1)^{2J_3} J_3^2. \tag{1.6}$$

Here J_3 is the 3-component of spatial angular momentum. Since every BPS particle in an $\mathcal{N}=2$ theory has a universal half-hypermultiplet factor $(\mathbf{0},\mathbf{0};\mathbf{\tfrac{1}{2}})$ obtained from quantizing the fermionic degrees of freedom associated to its center of mass in $\mathbb{R}^3$, we can also write $\mathcal{H}(p,q;t) = (\mathbf{0},\mathbf{0};\mathbf{\tfrac{1}{2}}) \otimes \mathcal{H}'(p,q;t)$ and

$$\Omega(p,q;t) = -2\,\mathrm{Tr}_{\mathcal{H}'(p,q;t)}\left(2(J_3')^2 - (J_3'-\frac{1}{2})^2 - (J_3'+\frac{1}{2})^2\right)(-1)^{2J_3'} = \mathrm{Tr}_{\mathcal{H}'(p,q;t)}\,(-1)^{2J_3'}. \tag{1.7}$$

Here J_3' is the reduced angular momentum.

We include the t-dependence since even though Ω is an index, it *does* depend on the background complexified Kähler moduli t^A, through jumping phenomena at walls of marginal stability. These are walls where the phases of the central charges of the constituents of a bound state line up, so decay into them is no longer energetically obstructed. As we will see this is not just a minor nuisance; it affects the regime of validity of the OSV conjecture in a significant way and moreover associated wall crossing formulae for the index will prove a powerful tool, central in our derivation.

With these indices one can define a (formal) partition sum at fixed magnetic charge p by summing over electric charges q:[7]

$$\mathcal{Z}_{\rm BH}(\phi;t_\infty) := \sum_q \Omega(p,q;t_\infty)\, e^{2\pi\phi^\Lambda q_\Lambda}. \tag{1.8}$$

In terms of this generating function the conjecture [10] states that in a suitable parameter regime,

$$\mathcal{Z}_{\rm BH}(\phi;t_\infty) \sim |\mathcal{Z}_{\rm top}(g_{\rm top},t)|^2 \tag{1.9}$$

[5] Sometimes q refers just to the $D2$ charge and not the total electric charge. In this case it should be clear from context which one is meant.

[6] The normalization factor -2 is chosen such that one (half) hypermultiplet gives $\Omega = +1$.

[7] Our normalization conventions differ from [10, 11, 12]: $\phi(\text{here}) = -\phi([10])/2\pi = -\phi([11, 12])/2$.

where $\mathcal{Z}_{\rm top}$ is the topological string partition function and the following substitutions are understood:

$$g_{\rm top} = \frac{4\pi i}{X^0} = \frac{4\pi}{2I^0{}_\Lambda \phi^\Lambda + i\, p^0}, \qquad t^A = \frac{2I^A{}_\Lambda \phi^\Lambda + i\, p^A}{2I^0{}_\Lambda \phi^\Lambda + i\, p^0}. \tag{1.10}$$

Here $I^{\Lambda_1}{}_{\Lambda_2}$ is the inverse symplectic intersection form between magnetic and electric charges, where intersection products are assumed to equal zero between magnetic charges and between electric charges. The presence of this intersection form can be deduced for example from the results of [24]. In a canonical symplectic charge basis (which was assumed in [10]), one has $I^{\Lambda_1}{}_{\Lambda_2} = \delta^{\Lambda_1}_{\Lambda_2}$, but more generally it is sometimes more natural to work in a basis with a different intersection form. For example, as we review in appendix A, in type IIA at large radius, where RR charges are given by $\Gamma = {\rm ch}(F) \wedge \sqrt{\hat{A}}$, the natural choice of basis gives $I^{\Lambda_1}{}_{\Lambda_2} = \sigma_{\Lambda_2}\, \delta^{\Lambda_1}_{\Lambda_2}$ where $\sigma_0 = 1$, $\sigma_A = -1$. We trust the reader will not confuse the background moduli t_∞ with the t^A substituted on the RHS of the OSV conjecture.

As mentioned already in the introduction, an alternative way of writing (1.9) is

$$\Omega(p,q;t_\infty) \sim \int d\phi\, e^{-2\pi \phi^\Lambda q_\Lambda}\, |\mathcal{Z}_{\rm top}|^2, \tag{1.11}$$

where again the substitutions (1.10) are understood on the right hand side.

In this text, we will define the topological string partition function $\mathcal{Z}_{\rm top}$ associated to a Calabi-Yau threefold X as follows:

$$\mathcal{Z}_{\rm top}(g,t) := \mathcal{Z}_{\rm pol}(g,t)\, \mathcal{Z}^0_{GW}(g)\, \mathcal{Z}'_{\rm GW}(g,t) \tag{1.12}$$

$$\mathcal{Z}_{\rm pol}(g,t) := \exp\biggl(-\frac{(2\pi i)^3}{6g^2} D_{ABC} t^A t^B t^C - \frac{2\pi i}{24} c_{2A} t^A \biggr) \tag{1.13}$$

$$\mathcal{Z}^0_{GW}(g) := \biggl(\prod_n (1 - e^{-gn})^n \biggr)^{-\chi(X)/2} \tag{1.14}$$

$$\mathcal{Z}'_{GW}(g,t) := \exp\biggl(\sum_{\beta \neq 0} \sum_h N_{h,\beta}\, (-g^2)^{h-1}\, e^{2\pi i \beta_A t^A} \biggr). \tag{1.15}$$

Here D_{ABC} are the triple intersection numbers of the basis $\{D_A\}_A$, c_2 is the second Chern class of X, $\chi(X)$ is the Euler characteristic of X, and $N_{h,\beta}$ are the Gromov-Witten invariants counting the "number" of holomorphic maps of genus h into class $\beta \in H_2(X,\mathbb{Z})$. Henceforth we will usually drop the subscript on $g_{\rm top}$ and simply write g.

In (1.14), we resummed the contributions of the degree zero Gromov-Witten invariants $N_{h,0}$ into the McMahon form (1.14). At small g, it is more suitable to use the asymptotic expansion given by

$$\mathcal{Z}^0_{GW}(g) \approx K \left(\frac{g}{2\pi} \right)^{\frac{\chi(X)}{24}} \exp\left(\frac{\chi(X)}{2} \frac{\zeta(3)}{g^2} + \sum_{h=1}^{\infty} N_{h,0}\, (-g^2)^{h-1} \right). \tag{1.16}$$

where K is a constant. (See Appendix E of [12], or eq.(4.34) et. seq. of [41] for a careful derivation.)

Similarly, one can rewrite $\mathcal{Z}'_{GW}(g,t)$ as an infinite product using M2 BPS invariants [8] [32, 33, 34]. Denoting the BPS invariants as n^r_q we have $\mathcal{Z}'_{GW}(g,t) = \mathcal{Z}'_{GV}(e^{-g}, e^{2\pi i t})$, where

$$\mathcal{Z}'_{GV}(e^{-g}, e^{2\pi i t}) = \prod_{q>0,k>0} (1 - e^{-gk+2\pi i q\cdot t})^{kn^0_q} \tag{1.17}$$

$$\times \prod_{q>0,r>0} \prod_{\ell=0}^{2r-2} \left(1 - e^{-g(r-\ell-1)+2\pi i q\cdot t}\right)^{(-1)^{r+\ell}\binom{2r-2}{\ell} n^r_q}. \tag{1.18}$$

Finally, $\mathcal{Z}_{\rm top}$ is conjecturally related [35, 36, 37, 29] to the Donaldson-Thomas partition function for ideal sheaves, as

$$\mathcal{Z}'_{DT}(u,v) = \mathcal{Z}'_{GV}(-u,v). \tag{1.19}$$

Here

$$\mathcal{Z}_{DT}(u,v) := \sum_{n,\beta} N_{DT}(\beta,n)\, u^n\, v^\beta := \mathcal{Z}^0_{DT}(u)\, \mathcal{Z}'_{DT}(u,v) \tag{1.20}$$

$$\mathcal{Z}^0_{DT}(u) := \prod_n (1-(-u)^n)^{-n\chi(X)} = \left(\mathcal{Z}^0_{\rm GW}\right)^2, \tag{1.21}$$

where $v^\beta := \prod_A (v_A)^{\beta_A}$, and $N_{DT}(\beta,n)$ are the ideal sheaf DT invariants, defined in [38, 39, 40]. Physically they can be thought of as counting D6-D2-D0 BPS states with D0-charge n and D2-charge $-\beta$, *ignoring* stability

(i.e. ignoring D-term constraints on the D6-D2-D0 moduli space).

The conjecture (1.19) has been confirmed by many case studies, partially proved, and is physically well supported [35, 29]. We will assume it is true.

We conclude with two remarks:

1. Our definition of $\mathcal{Z}_{\rm top}$ is slightly nonstandard because of the way we handled the $\beta = 0$ invariants. From (1.16) we see that there is an extra summand $\frac{\chi}{24}\log\frac{g}{2\pi}$ in the definition of $F_{\rm top}$. This is large for g both small and large, and has the important property that the expansion of $F_{\rm top}$ is *not* analytic in g.

2. For many Calabi-Yau manifolds one can use known asymptotic growth estimates of the Gromov-Witten invariants following from the results of [116, 117] to show that the first line of (1.17) indeed converges as an analytic product for sufficiently small u and sufficiently large Kähler classes. It will, however, have interesting singularities. On the other hand, it already follows from the results of [34] that $\mathcal{Z}^{r>0}_{DT}$ has zero radius of convergence, and must be considered a formal product. In section 6.1 we will give a nice physical interpretation of this second product as a product over "core states."

[8] These are also known as Gopakumar-Vafa invariants.

2. A fareytail expansion for the D4-D2-D0 partition function

In this section we will show that $\mathcal{Z}_{\rm BH}$ for $p^0 = 0$ and $t_\infty = i\infty$ can in general be expressed as a fareytail (or Rademacher-Jacobi) series built from its polar part, generalizing the results of [26, 27]. The derivation given here is dual to the derivations [18, 28] which appeared while this paper was being written. We include it nevertheless for completeness and because we fill in some of the gaps in those proofs and clarify some issues which were left open, e.g. how to define a fareytail series for the actual partition function instead of for the fareytail transform of it.

2.1 D-brane model, physical interpretation, and S-duality

Consider a single D4-brane wrapped on a smooth holomorphic surface in X. This surface will be in an ample divisor class $P = P^A D_A$ and we often (somewhat sloppily) refer to the surface also as P. We assume the surface has N $\overline{D0}$-branes[9] bound to it and $U(1)$ flux $F \in H^2(P)$ turned on.[10]

Defining electric charges as the quantities coupling to the RR-potentials, the D2-brane charges are

$$q_A = Q_A = D_A \cdot F. \tag{2.1}$$

Here and in what follows the scalar product between two 2-forms in $H^2(P)$ is the intersection product:

$$\alpha_1 \cdot \alpha_2 := \int_P \alpha_1 \wedge \alpha_2 \tag{2.2}$$

If the 2-forms are pulled back from $H^2(X)$, i.e. $\alpha_i = \iota_P^* \hat{\alpha}_i$, this can also be written as

$$\alpha_1 \cdot \alpha_2 = P \cdot \hat{\alpha}_1 \cdot \hat{\alpha}_2 = D_{ABC}\, P^A \hat{\alpha}_1^B \hat{\alpha}_2^C. \tag{2.3}$$

To avoid cluttering, in the following we usually will not notationally distinguish between $\hat{\alpha}_i$ and its pullback α_i, hoping that this will not cause confusion.

The total D0-brane charge is

$$q_0 = -N + \frac{1}{2} F^2 + \frac{\chi(P)}{24} \tag{2.4}$$

where

$$\chi(P) = P^3 + c_2(X) \cdot P := D_{ABC} P^A P^B P^C + c_{2,A} P^A \tag{2.5}$$

is the Euler characteristic of P. The last term in (2.4) represents the curvature induced D0-brane charge on the D4-brane.

To have a supersymmetric configuration, one needs $F^{(2,0)} = 0 (= F^{(0,2)})$. (The B-field does not appear here because it is always of $(1,1)$ type for flat B.) For generic fluxes at generic points in the D4-brane moduli space, this condition will not be satisfied. Exceptions are fluxes F which are pulled back from $H^2(X) = H^{1,1}(X)$: for these, $F^{(2,0)} = 0$ identically.

[9] In our conventions, which follow [42] and are natural from a geometric point of view, D4 branes form bound states with *anti*-D0 branes at large radius, and D6-branes with anti-D2 branes.

[10] $H^2(P)$ and $H^2(X)$ will refer to the *integral* cohomology modulo its torsion subgroup.

However, in general there will be (many) elements of $H^2(P)$ which are not pulled back from $H^2(X)$. For these, the condition $F^{(2,0)} = 0$ imposes $h^{2,0}(P)$ equations on the $h^{2,0}(P)$ geometric moduli of P, which will generically restrict the divisor moduli to a set of isolated points, and more generally to a subvariety of the original moduli space [43, 44]. The N $\overline{D}0$-branes bound to the D4 are pointlike[11] and not obstructed by the fluxes.

Thus we can rewrite (1.8) as

$$\mathcal{Z}_{\mathrm{BH}}(\phi^0, \Phi) = \sum_{F,N} d(F,N)\, e^{2\pi\phi^0[-N+\frac{1}{2}F^2+\frac{\chi(P)}{24}]+2\pi\Phi\cdot F} \tag{2.6}$$

where the sum is over $U(1)$ worldvolume fluxes F on P and the number $N \geq 0$ of bound pointlike anti-D0 branes. The flux lattice is $L = \sigma/2 + H^2(P)$, with $\sigma/2 := c_1(P)/2 = \iota_P^* P/2 \bmod 1$, where ι_P is the embedding map of the surface P. We will choose the natural representative $\sigma = \iota_P^* P$, which we will also denote simply as P. The half-integral shift follows from the K-theoretic formulation of RR charges and is needed to cancel anomalies, both on the brane worldvolume [45] and on the fundamental string worldsheet [46].

The index $d(F,N)$ is defined similarly to (1.6)-(1.7) but now with the trace in a sector of fixed (F,N), for $J \to \infty$. The angular momentum can be identified with the Lefshetz $SU(2)$ action on the moduli space, so in particular the 3-component of the spin of a p-form is $J_3' = (p - \dim)/2$, where dim is the complex dimension of moduli space [91]. This identifies our index $d(F,N)$ up to a sign with the Euler characteristic of the moduli space $\mathcal{M}_{F,N}$ of BPS configurations in the sector labeled by (F,N):

$$d(F,N) = (-1)^{\dim \mathcal{M}_{F,N}}\, \chi(\mathcal{M}_{F,N}). \tag{2.7}$$

Note that $d(F,N)$ is independent of the B-field, because B does not appear in the (large radius) BPS conditions for D4-D2-D0 bound states and does not affect the moduli spaces.

Typically $\mathcal{M}_{F,N}$ has singularities, and as a result it is not directly clear what the proper mathematical definition is of the Euler characteristic $\chi(\mathcal{M}_{F,N})$ to get the correct physical index. It would be worthwhile to have a precise mathematical definition of the invariants $d(F,N)$. Quite possibly they are DT invariants for torsion sheaves [38, 39, 40]. Some discussion of the subtleties involved can be found in [47].

It was noted e.g. in section 6 of [12] that this partition function is everywhere divergent, but that this can be cured in a natural way by adding a Boltzmann weight $e^{-\beta H}$ with H the BPS energy of the state. The resulting partition sum is then naturally interpreted as the BPS partition function[12] of a single D4-brane wrapping P and a Euclidean time circle of circumference β, in a background with flat RR potentials[13]

$$C_3 =: C \wedge \frac{dt}{\beta}, \qquad C_1 =: C_0 \wedge \frac{dt}{\beta}. \tag{2.8}$$

[11] When a suitable nonzero B-field is turned on, they can alternatively be considered to be smooth noncommutative $U(1)$ instantons.

[12] This should be given by the partition function of a suitably topologically twisted D4 DBI theory, possibly the theory constructed in [48]. It would be interesting to make this precise.

[13] Flat RR potentials are properly described by the compact K-group $K^{-1}(X;\mathbb{R}/\mathbb{Z})$. This determines the proper periodicities for these fields. We will ignore such subtleties in this paper.

Here $C \in H^2(X,\mathbb{C})$. The BPS partition function is roughly $\mathrm{Tr}\,(-1)^{2J_3'}\, e^{-\beta H - 2\pi i q_\Lambda \cdot C^\Lambda}$ where the trace sums over all D4 states including all sectors (F,N). More precisely (in units with $\ell_s := 2\pi\sqrt{\alpha'} = 1$):

$$\mathcal{Z}_{D4}(\frac{\beta}{g_{\rm IIA}}, C_0, C; B+iJ) := \mathrm{Tr}\,(-1)^{2J_3'} e^{-\beta H - 2\pi i[-N+\frac{1}{2}\mathcal{F}^2+\frac{\chi(P)}{24}]C_0 - 2\pi i \mathcal{F}\cdot(C+\frac{P}{2})} \tag{2.9}$$

$$= \sum_{F,N} d(F,N)\, e^{-\frac{2\pi\beta}{g_{\rm IIA}}|Z(F,N;B+iJ)|}\, \times$$

$$\times\, e^{-2\pi i[-N+\frac{1}{2}\mathcal{F}^2+\frac{\chi(P)}{24}]C_0 - 2\pi i \mathcal{F}\cdot(C+\frac{P}{2})} \tag{2.10}$$

where $\mathcal{F} := F - B$. The "extra" factor $e^{-\pi i \mathcal{F}\cdot P}$ must be there for S-duality to work properly [49, 50, 51], as we will confirm below. The string coupling constant $g_{\rm IIA}$ is the physical IIA coupling, not to be confused with the topological string coupling. The quantity $Z(F,N;B+iJ)$ denotes the holomorphic central charge of the D4-D2-D0 system, which in our conventions with charge $\Gamma \in H^{\rm even}(X)$ at large J is given by (A.8):

$$Z(F,N;B+iJ) = -\int_X e^{-(B+iJ)}\Gamma = \frac{1}{2}J^2 + i\mathcal{F}\cdot J + [N - \frac{1}{2}\mathcal{F}^2 - \frac{\chi(P)}{24}]. \tag{2.11}$$

Recall that $J^2/2 \equiv \int_P J^2/2$ is the volume of P. The absolute value of Z is proportional to the DBI energy evaluated on BPS configurations. Note that equation (2.9) does depend on β and the background metric, but in a quasi-topological way, fixed by charges and background Kähler moduli.

In the limit $J \to \infty$ we have

$$|Z| = \frac{1}{2}J^2 + \frac{(\mathcal{F}\cdot J)^2}{J^2} + [N - \frac{1}{2}\mathcal{F}^2 - \frac{\chi(P)}{24}] + O(1/J^2). \tag{2.12}$$

Since $\mathcal{F}$ is of type $(1,1)$ on supersymmetric solutions, we can use the Hodge index theorem, which states that the lattice of $(1,1)$ classes has Lorentzian signature, to decompose $\mathcal{F}$ in self-dual and anti-selfdual parts as:

$$\mathcal{F} = \mathcal{F}_+ + \mathcal{F}_-, \qquad \mathcal{F}_+ = \frac{\mathcal{F}\cdot J}{J^2}\, J, \qquad \mathcal{F}_- \cdot J = 0. \tag{2.13}$$

We refer to appendix B for more details. With this we can redefine $\mathcal{Z}_{D4}$, dropping an irrelevant (S-duality invariant) overall factor $e^{-2\pi {\rm Im}\tau {\rm Vol}}$ as:

$$\mathcal{Z}_{D4}(\tau, C, B) := \sum_{F,N} d(F,N)\, e^{2\pi i\tau[N-\frac{1}{2}\mathcal{F}_-^2 - \frac{\chi(P)}{24}] - 2\pi i\bar\tau\frac{1}{2}\mathcal{F}_+^2 - 2\pi i\mathcal{F}\cdot(C+\frac{P}{2})}, \tag{2.14}$$

where

$$\tau = C_0 + \frac{\beta}{g_{\rm IIA}}\, i. \tag{2.15}$$

Since $\mathcal{F}_-^2 < 0$ and $\mathcal{F}_+^2 > 0$, the sum over fluxes is well behaved in (2.14).

Alternatively we can write (2.9) as

$$\mathcal{Z}_{D4} = e^{-2\pi i\tau\frac{\chi(P)}{24}} \sum_{F,N} d(F,N)\, e^{-2\pi[{\rm Im}\tau\int\frac{1}{2}\mathcal{F}\wedge *\mathcal{F} + i\,{\rm Re}\tau\int\frac{1}{2}\mathcal{F}\wedge\mathcal{F} - i\tau N + i\mathcal{F}\cdot(C+\frac{P}{2})]}. \tag{2.16}$$

Not surprisingly, part of the exponent has the form of a $U(1)$ Yang-Mills energy with complexified coupling constant τ. Nevertheless, the coefficients $d(F,N)$ are nontrivial and the expression is not proportional to the standard theta function of a $U(1)$ gauge theory. The reason for this is that the theory we are considering is more than just standard topologically twisted $\mathcal{N}=4$ $U(1)$ Yang-Mills, since we consider arbitrarily large deformations of the D4-brane, which are only properly described by the full DBI theory. Moreover, we include pointlike bound $\overline{D0}$-branes, which do not correspond to standard smooth Yang-Mills instantons. Note also that the curvature term proportional to $\chi(P)$, which is crucial for modular invariance, appears naturally here.

The OSV black hole partition function $\mathcal{Z}_{\rm BH}$ is obtained by making a formal substitution of arguments in $\mathcal{Z}_{D4}$: [14]

$$\mathcal{Z}_{\rm BH}(\phi^0,\phi^A) := \mathcal{Z}_{\rm D4}(\beta=0, B=0, C_0=i\phi^0, C=i\Phi-\frac{P}{2}). \tag{2.17}$$

We put $B=0$ because we do not want to introduce explicit B-dependence in $\mathcal{Z}_{\rm BH}$. Here ϕ^0,Φ are real.

Unlike the partition function $\mathcal{Z}_{D4}$, which from (2.14) and the large N asymptotics $d(F,N)\sim e^{k\sqrt{N}}$ [2, 3] is easily seen to converge for any $\beta>0$, the partition function $\mathcal{Z}_{\rm BH}$ diverges everywhere. We can nevertheless make sense of it and justify formal manipulations by turning on β at intermediate steps. For example we can write

$$\Omega(p,q) = \lim_{\beta\to 0}\oint dC_0\, dC\, \mathcal{Z}_{D4}(C_0+\tfrac{\beta}{g_{\rm IIA}}i, C, B=0)\, e^{2\pi i q_0 C_0 + 2\pi i Q\cdot(C+\frac{P}{2})}. \tag{2.18}$$

The integrals run over one period of all RR potentials. [15] They are well defined, and produce $\Omega(p,q)\, e^{-\beta|Z|/g}$, which in the limit $\beta\to 0$ reduces to $\Omega(p,q)$. Often however it is not necessary to perform the regularization explicitly; for example if one is only interested in a saddle point evaluation of the integral, one can proceed formally.

Besides providing a good regularization, this "physical" interpretation of the OSV partition function also allows applying the usual T- and S-dualities one expects to be symmetries of $\mathcal{Z}_{D4}$. In particular performing a TST duality transforms this into a form getting significantly closer to what one needs to derive the conjecture.

The T-duality goes along the time circle, and trivially preserves $\mathcal{Z}$ but gives it the interpretation of a partition sum over supersymmetric configurations of a Euclidean D3-brane wrapped on P, with Euclidean time circumference $1/\beta$, IIB coupling $g_{\rm IIB}=g_{\rm IIA}/\beta$ and RR potentials C_0 and C.

Next, we S-dualize. The D3-brane is self-dual under S-duality [52], which maps $\tau = C_0 + i/g_{\rm IIB}$ to $-1/\tau$ while acting as electric-magnetic duality on the $U(1)$ gauge fields. The background fields transform as

$$\tau' = -1/\tau, \qquad C' = -B, \quad B' = C, \quad J' = \sqrt{C_0^2 + g_{\rm IIB}{}^{-2}}\, J. \tag{2.19}$$

[14] This is sometimes referred to as the "OSV limit." However it is not in any sense a well-defined limit.

[15] Again, the K-theoretic interpretation can modify the proper periods. This will at most result in a modest numerical factor in 2.18. We will ignore this possibility.

Note that the transformation of the Kähler form J leaves the background $J = \infty$ we are considering invariant. The partition function must transform as a modular form with some weights $(w, \bar{w})$, that is

$$\mathcal{Z}'_{D3} = \omega_S \, \tau^w \bar{\tau}^{\bar{w}} \mathcal{Z}_{D3}. \tag{2.20}$$

where ω_S is a phase and, for fractional $w, \bar{w}$ we use the principal branch of the logarithm. The sum over fluxes F in $\mathcal{Z}'_{D3}$ is now over the dual lattice, but since $H^2(P)$ is self-dual on a compact surface, this is the same as the original lattice. In examples which can be checked explicitly this equality of partition sums essentially amounts to a Poisson resummation (see also appendix G). Finally we can do another T-duality along the time circle to go back to IIA, but this is again trivial.

We thus have

$$\begin{aligned}
\mathcal{Z}(\tau, C) &:= \mathcal{Z}_{D4}(\tau, C, B = 0) && (2.21)\\
&= \omega_S^{-1} \tau^{-w} \bar{\tau}^{-\bar{w}} \, \mathcal{Z}_{D4}(-\frac{1}{\tau}, 0, C) && (2.22)\\
&= \omega_S^{-1} \tau^{-w} \bar{\tau}^{-\bar{w}} \sum_{F,N} d(F,N) \, e^{\frac{2\pi i}{\tau}[-N + \frac{1}{2}(F_- - C_-)^2 + \frac{\chi(P)}{24}] + \frac{\pi i}{\bar{\tau}}(F_+ - C_+)^2 - \pi i F \cdot P} && (2.23)\\
&= \omega_S^{-1} \tau^{-w} \bar{\tau}^{-\bar{w}} \, e^{\pi i (\frac{1}{\tau} C_-^2 + \frac{1}{\bar{\tau}} C_+^2)} && (2.24)\\
&\qquad \times \sum_{F,N} d(F,N) \, e^{\frac{2\pi i}{\tau}[-N + \frac{1}{2} F_-^2 + \frac{\chi(P)}{24}] + \frac{\pi i}{\bar{\tau}} F_+^2 - 2\pi i (F_- \cdot \frac{C}{\tau} + F_+ \cdot \frac{C}{\bar{\tau}}) - \pi i F \cdot P} && (2.25)\\
&= \omega_S^{-1} \tau^{-w} \bar{\tau}^{-\bar{w}} \, e^{\pi i (\frac{1}{\tau} C_-^2 + \frac{1}{\bar{\tau}} C_+^2)} \, \mathcal{Z}_{D4}(-\frac{1}{\tau}, \frac{C}{\tau}, B = 0) && (2.26)\\
&= \omega_S^{-1} \tau^{-w} \bar{\tau}^{-\bar{w}} \, E[-\frac{C^2}{2\tau}] \, \mathcal{Z}(-\frac{1}{\tau}, \frac{C}{\tau}). && (2.27)
\end{aligned}$$

In the last line we introduced the shorthand notation

$$E[f(\tau) X \cdot Y] := e^{-2\pi i \, f(\tau) \, X_- \cdot Y_- - 2\pi i \, f(\bar{\tau}) \, X_+ \cdot Y_+}, \qquad E[A + B] := E[A] \, E[B]. \tag{2.28}$$

To summarize, if we define

$$\mathcal{Z}(\tau, C) := \sum_{F,N} d(F,N) \, e^{2\pi i \tau [N - \frac{1}{2}(F_-)^2 - \frac{\chi(P)}{24}] - 2\pi i \bar{\tau} \frac{1}{2}(F_+)^2 - 2\pi i F \cdot (C + \frac{P}{2})} \tag{2.29}$$

with $F \in H^2(P) + P/2$ then we have the following modular representation. For $A \in \Gamma := SL(2, \mathbb{Z})$ denote

$$A \cdot (\tau, C_+, C_-) := (\frac{a\tau + b}{c\tau + d}, \frac{C_+}{c\bar{\tau} + d}, \frac{C_-}{c\tau + d}) \tag{2.30}$$

(Sometimes we will abbreviate this equation to $A \cdot (\tau, C) = (\frac{a\tau+b}{c\tau+d}, \frac{C}{c\tau+d})$. Also, note that this action does not factor through $PSL(2, \mathbb{Z})$, indeed, $S^2 \cdot (\tau, C) = (\tau, -C)$.) Then

$$\mathcal{Z}(A \cdot (\tau, C)) = \omega_A \, (c\tau + d)^w (c\bar{\tau} + d)^{\bar{w}} \, E[\frac{c}{c\tau + d} \frac{C^2}{2}] \, \mathcal{Z}(\tau, C). \tag{2.31}$$

Here ω_A is a phase depending on the $SL(2, \mathbb{Z})$ element. For

$$A = T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \tag{2.32}$$

direct computation leads to

$$\omega_T = e^{-2\pi i \frac{P^3}{8} - 2\pi i \frac{\chi}{24}} = e^{2\pi i \frac{c_2 \cdot P}{24}} \tag{2.33}$$

In the second equality we used the index theorem which says that

$$I_P := \frac{P^3}{6} + \frac{c_2 \cdot P}{12} \tag{2.34}$$

is an integer. To verify consistency of the modular representation of

$$A = S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{2.35}$$

we need to use $d(-F, N) = d(F, N)$. In this case consistency of the modular representation and the theta function decomposition described below leads to

$$\omega_S = -e^{i\pi I_P} e^{i\frac{\pi}{2}(\bar{w} - w)} e^{i\frac{\pi}{2}P^2} \tag{2.36}$$

We will find later that $\bar{w} = 1/2$ and, for $b_1(X) = 0$, $w = -3/2$. Using this one checks that indeed $\omega_T^3 = \omega_S^{-1}$. We will not need an explicit formula for ω_A for all A.

What we are ultimately interested in are saddle point evaluations of integrals like (2.18). For saddle points at small τ, which arise for large Q_0, the resummed expansion (2.25) is particularly useful, since when $\tau \to 0$, the subleading terms in this expression are exponentially suppressed. We will make this more precise in section 2.3.

2.2 Theta function decomposition

It it is useful and instructive to decompose $\mathcal{Z}$ as a sum of theta functions. To do this, we decompose F in parts according to the distinction between fluxes which are pulled back from X, and fluxes orthogonal to these. As before, let ι_P be the embedding map for our divisor P. Then $L_X := \iota_P^* H^2(X)$ is the lattice of fluxes pulled back from the ambient space X, a basis of which is formed by $\iota_P^* D_A$. This has metric $D_{AB} := D_{ABC}P^C$. Because in general $\det D_{AB} \neq 1$, the lattice is not unimodular, while $H^2(P)$ is. This implies that the lattice $L_X \oplus L_X^\perp$ is only a sublattice of $H^2(P)$. The quotient $\mathcal{D}$ of the latter by the former is a finite group, parametrized by "glue vectors" $\gamma \in \mathcal{D}$. Taking into account the half-integral shift $P/2$ of the flux mentioned earlier, we thus get the following decomposition for fluxes $F \in H^2(P)$:

$$F = \frac{P}{2} + f^\parallel + \gamma + f^\perp, \tag{2.37}$$

where $f^\parallel \in L_X$, $f^\perp \in L_X^\perp$. We can further decompose γ along $\mathbb{Q} \otimes L_X$ and its orthogonal complement $\mathbb{Q} \otimes L_X^\perp$:

$$\gamma = \gamma^\parallel + \gamma^\perp. \tag{2.38}$$

Any nontrivial γ must have simultaneously $\gamma^\parallel \neq 0$ and $\gamma^\perp \neq 0$. That it must have a nonzero $\gamma^\parallel$ is clear: otherwise γ is an integral flux orthogonal to L_X, which by definition is in $L_X^\perp$ and hence trivial in $\mathcal{D}$. That it must have nonzero $\gamma^\perp$ as well is more subtle. If

$\gamma^\perp = 0$, then we can write $\gamma = r^A \iota_P^* D_A$. Now because P is very ample, by the Lefshetz hyperplane theorem (see appendix B), the map $\iota_P : H_2(P,\mathbb{Z}) \to H_2(X,\mathbb{Z})$ is surjective, that is, every 2-cycle in X can be realized as a 2-cycle in P. Hence there is in particular a set σ^A of 2-cycles on P such that $\iota_P(\sigma^A)$ is a basis of $H_2(X,\mathbb{Z})$ dual to the D_A. Because γ is integral, we moreover have $\int_{\sigma^A} \gamma \in \mathbb{Z}$. But by construction, this equals r^A. Therefore $\gamma \in L_X$, so it is trivial as an element of $\mathcal{D}$.

We can also identify $\mathcal{D}$ with the discriminant group: $\mathcal{D} = L_X^*/L_X$. Since $H^2(P)$ is unimodular, the embedding is specified by an isomorphism with $(L_X^\perp)^*/L_X^\perp$ preserving quadratic forms, by the Nikulin primitive embedding theorem [53] (the embedding is primitive again because of the Lefshetz hyperplane theorem). Similarly, $\gamma^\perp \in (L_X^\perp)^*$ represents γ under this isomorphism.

Note that $d(F,N)$ does not depend on the L_X part of F, since this is automatically of type $(1,1)$ and hence does not affect the BPS condition or the moduli space of supersymmetric configurations. Using this, the partition function (2.29) can be written as [16]

$$\mathcal{Z}(\tau, C) = \sum_\gamma \Psi_\gamma(\tau,\bar\tau,C)\, H_\gamma(\tau). \tag{2.39}$$

Here we defined, using the shorthand notation (2.28)

$$\Psi_\gamma(\tau,\bar\tau,C) := \sum_{f^\parallel} E[\frac{\tau}{2}(\frac{P}{2} + \gamma^\parallel + f^\parallel)^2 + (\frac{P}{2} + \gamma^\parallel + f^\parallel)\cdot(C + \frac{P}{2})], \tag{2.40}$$

which is a nonholomorphic Siegel-Narain theta function of signature $(1, h-1)$, implicitly depending on the Kähler form J because this determines the (C_+, C_-)-split. We furthermore defined the holomorphic

$$H_\gamma(\tau) := \sum_{f^\perp, N} d(\frac{P}{2} + \gamma + f^\perp, N)\, e^{-2\pi i \tau \hat{q}_0(F,N)} \tag{2.41}$$

where

$$\widehat{q}_0(F,N) = \frac{\chi(P)}{24} + \frac{1}{2}(f^\perp + \gamma^\perp)^2 - N = q_0 - \frac{1}{2}(\frac{P}{2} + \gamma^\parallel + f^\parallel)^2 = q_0 - \frac{Q^2}{2} \tag{2.42}$$

Note that $H_\gamma(\tau) = H_{-\gamma}(\tau)$.

All nontrivial information about the degeneracies is captured by the holomorphic $H_\gamma(\tau)$. For example we have for the degeneracies [17]

$$\Omega(P,Q,q_0) = \oint d\tau H_{\gamma_Q}(\tau)\, e^{2\pi i \tau \widehat{q}_0} \tag{2.43}$$

[16] This decomposition can be understood in the AdS/CFT correspondence as the decomposition of the partition function obtained by factoring out the singleton modes, as in [54]. The analogous singleton decomposition for the M5-brane partition function was used in [18]. The general singleton decomposition of the M5-brane partition function was described in [122].

[17] We abbreviate $\Omega(0,P,Q,q_0)$ by $\Omega(P,Q,q_0)$, and of course $t_\infty = i\infty$ is understood.

where γ_Q is uniquely determined by $(\gamma_Q)_A = Q_A - D_{ABC}P^BP^C/2 \mod D_{ABC}P^Bn^C$, $n^C \in \mathbb{Z}$, and

$$\widehat{q}_0 := q_0 - \frac{Q^2}{2}, \qquad Q^2 := D^{AB}Q_AQ_B. \tag{2.44}$$

The proof of (2.43) proceeds as follows: First, note that fixing the D2-charge Q_A fixes $D_A \cdot F$, which puts $P/2 + \gamma^{\|} + f^{\|} = Q$. This determines γ (and $f^{\|}$) uniquely as stated above, because the difference of two different γ's satisfying this equation would give a nontrivial element of $\mathcal{D}$ with zero $\|$-component, which as we saw does not exist. Put differently, for each γ and $\hat{q}_0$ we have an equivalence class

$$[\gamma, \hat{q}_0] := \{(0, P, Q, q_0) \,|\, q_0 - \frac{Q^2}{2} = \hat{q}_0 \quad \text{and} \quad Q \in \iota_{P,*}(L_X + \frac{P}{2} + \gamma)\}. \tag{2.45}$$

As noted above, shifts of F by elements of L_X do not change the index of BPS states, hence the index $\Omega([\gamma, \hat{q}_0]) := \Omega(0, P, Q, q_0)$ only depends on the equivalence class $[\gamma, \hat{q}_0]$. Another way of phrasing this is that the D4-D2-D0 BPS spectrum at $J \to \infty$ is invariant under integral B-shift monodromy, in accord with the absence of walls of marginal stability running off to $J = \infty$ for this system.

Grouping terms in (2.41) with fixed $\hat{q}_0$ we can thus also write

$$H_\gamma(\tau) := \sum_{\hat{q}_0} \Omega([\gamma, \hat{q}_0])\, e^{-2\pi i \tau \hat{q}_0}. \tag{2.46}$$

Now, using the S-duality transformation (2.27) of $\mathcal{Z}$ and integrating both sides with respect to C ranging over $H^2(X, \mathbb{R})$ one gets $\bar{w} = 1/2$ and

$$H_\gamma(\tau) = |\mathcal{D}|^{-1/2} (-i\tau)^{-w + \frac{h-1}{2}} (-1)^{I_P + 1} \sum_{\delta \in \mathcal{D}} e^{2\pi i \gamma^{\|} \cdot \delta^{\|}} H_\delta(-\frac{1}{\tau}). \tag{2.47}$$

where $|\mathcal{D}| = \#\mathcal{D} = \det(D_{ABC}P^C)$. Furthermore,

$$H_\gamma(\tau + n) = e^{-2\pi i n (\frac{(\gamma^\perp)^2}{2} + \frac{\chi}{24})}\, H_\gamma(\tau). \tag{2.48}$$

Thus we see that the H_γ form a modular vector. One can check consistency of the modular representation using the Gauss-Milgram sum formula [55]

$$\frac{1}{\sqrt{|\mathcal{D}|}} \sum_\gamma e^{-2\pi i \frac{1}{2} (\gamma^\perp)^2} = e^{-2\pi i \, \mathrm{sig}(L_X^\perp)/8}. \tag{2.49}$$

2.3 $\tau \to 0$ limit

When $\widehat{q}_0 \to -\infty$, the saddle point of (2.43) will be at $\tau \to 0$. To evaluate the integral, it is therefore useful to perform first the modular transformation (2.47). Indeed, when $\tau \to 0$, the only surviving term in the resummed series has $\gamma = N = f^\perp = 0$ since on supersymmetric configurations $(\gamma^\perp + f^\perp)^2 \leq 0$ with equality iff $f^\perp = 0$ and $\gamma^\perp = 0$, which as we saw in section (2.2) implies $\gamma = 0$. Hence in this limit

$$\Omega(P, Q, q_0) = d(\frac{P}{2}, 0)\, |\mathcal{D}|^{-1/2}\, (-1)^{I_P + 1} \oint d\tau\, e^{2\pi i \widehat{q}_0 \tau}\, (-i\tau)^{-w + \frac{h-1}{2}}\, e^{2\pi i \frac{\chi}{24\,\tau}}. \tag{2.50}$$

The saddle point of this integral lies at

$$\tau_* = i\sqrt{-\frac{\chi(P)}{24\widehat{q}_0}} \tag{2.51}$$

which is indeed small when $-\widehat{q}_0 \gg \chi(P)$, and

$$\ln\Omega(P,Q,q_0) = \frac{4\pi i\chi(P)}{\tau_*} = 2\pi\sqrt{-\frac{1}{6}\widehat{q}_0\,\chi(P)}. \tag{2.52}$$

Since $\chi(P) = P^3 + c_2\cdot P$, this reproduces the well-known result for the Bekenstein-Hawking-Wald entropy in this limit [2, 6]. One can do better however. The τ-integral can be done exactly, resulting in a Bessel function, as detailed in [11, 12]. This gives an explicit formula for $\Omega(P,Q,q_0)$ to all orders in a $1/\widehat{q}_0$ expansion, up to determination of w and $d(P/2,0)$.

In fact, comparison with an independent computation of $\mathcal{Z}_{\rm BH}$ in this regime using techniques developed in [30, 31] for counting closed string flux vacua, which we give in appendix G, fixes $w = -3/2$ for X a proper $SU(3)$ holonomy Calabi-Yau manifold.[18] Very roughly, the reason for this is that in the small ϕ^0 regime, the OSV partition function approximately factorizes in a factor $1/\eta(\phi^0)^{\chi(P)/24}$ counting the Euler characteristics of the D0-brane moduli spaces $\mathrm{Sym}^N P$, and a factor of the form $\int dF\, e^{c\phi^0 F^2}\cdots$, approximately counting flux vacua. The first factor gives a $(\phi^0)^{\chi(P)/2}$ after modular transformation, and the second factor a $(\phi^0)^{-b_2(P)/2}$ from integrating out F. Since $\chi(P) = b_2(P) + 2$ on an ample divisor in a proper Calabi-Yau, we thus get a net factor ϕ^0, corresponding to having $w + \bar{w} = -1$ in (2.27). We found in section 2.2 that $\bar{w} = 1/2$ (alternatively this can be directly deduced from the modular transformation properties of the theta functions Ψ_γ), hence $w = -3/2$ as claimed. We refer to appendix G for more details.

Furthermore, $d(P/2,0)$ is just the index of BPS states of the pure D4-brane without any deformation obstructing fluxes, which by (2.7) equals $(-1)^{\dim\mathcal{M}_P}\chi(\mathcal{M}_P)$ where $\mathcal{M}_P$ is the divisor deformation moduli space. It is not clear *a priori* what the physically relevant Euler characteristic of the divisor moduli space is, since this space has singularities where the divisor degenerates. It has an obvious compactification however, namely the corresponding linear system, which is the projectivization of the space of sections of the line bundle corresponding to P, which, because P is very ample, is just $\mathcal{M}_P = \mathbb{CP}^{I_P-1}$. Using this compactification, we thus have $d(P/2,0) = (-1)^{I_P-1}I_P$. Below we will give more evidence that this is the correct definition of $\chi(\mathcal{M}_P)$.

Rephrasing all of this in terms of the original OSV partition function, we conclude that for the purpose of computing $\widehat{q}_0 \to -\infty$ degeneracies, we can take

$$\mathcal{Z}(\tau,C) \approx (-1)^{I_P-1}I_P\,\omega_S^{-1}\tau^{3/2}\bar{\tau}^{-1/2}\,e^{2\pi i\frac{\chi(P)}{24\tau}}\,E[-\frac{C^2}{2\tau}]\,\Psi_0(-\frac{1}{\tau},\frac{C}{\tau}). \tag{2.53}$$

Making the OSV substitution $\tau = \bar{\tau} = i\phi^0$, and using (2.36), this formally becomes:

$$\mathcal{Z}_{\rm BH}(\phi^0,\Phi) \approx i\,I_P\,\phi^0 \sum_{S\in H^2(X,\mathbb{Z})} e^{\frac{2\pi}{\phi^0}[\frac{\chi(P)}{24}-\frac{1}{2}(\Phi+iS)^2]+\pi i P\cdot S}. \tag{2.54}$$

[18] More generally $w = -3/2 + b_1(X)$.

This is in rough agreement with the OSV formula (1.9), restricted to the polynomial part of the topological string partition function. The additional sum over shifts of Φ can be seen to be necessary to give the right hand side of (1.9) the same periodicity as the left hand side. In the integral formulation (1.11) of the conjecture, this sum can be traded for an extension of the periodic integration contours to the entire imaginary axis. A similar sum over shifts of ϕ^0 is absent here, but this is consistent with the small ϕ^0 approximation as the shifted terms are exponentially suppressed. We also find an additional measure factor $iI_P\phi^0$. Finally in this small ϕ^0 limit, the nonpolynomial corrections to $\mathcal{Z}_{\rm top}$ after the OSV substitutions are all exponentially small. Hence the above formula is in satisfactory agreement with the original OSV conjecture at small ϕ^0 (and in perfect agreement with our refinement of it which we will derive in the remainder of the paper).

2.4 A Rademacher-Jacobi formula

For larger values of τ, which is the regime relevant to the full OSV conjecture including instanton corrections, it is no longer sufficient to do a $\tau \to -1/\tau$ modular transformation to extract approximate expressions for the degeneracies, because the subleading terms in the q-expansion are no longer sufficiently suppressed in this limit to justify throwing them away. The key observation which will allow us to make progress is that because of its modular properties, $\mathcal{Z}$ can be entirely expressed in terms of a some kind of "$SL(2,\mathbb{Z})$ average" of a finite subset of terms, analogous to the Rademacher-Jacobi or fareytail expansion of [26].

At the end of section 2.2 we saw that H_γ transforms as a modular vector with weight[19]

$$w_H := w - \frac{h-1}{2} \tag{2.55}$$

The theta function vector Ψ_γ transforms in a conjugate way to ensure the transformation (2.31) of $\mathcal{Z}$. This can also verified directly using general properties of theta functions or by Poisson resummation. Thus, under general $SL(2,\mathbb{Z})$ transformations A, using the notation introduced above (2.31):

$$\mathcal{Z}(A\cdot(\tau,C)) = \omega_A\,(c\tau+d)^{w}(c\bar\tau+d)^{\frac{1}{2}}\,E[\frac{c}{c\tau+d}\frac{C^2}{2}]\,\mathcal{Z}(\tau,C) \tag{2.56}$$

$$\Psi_\gamma(A\cdot(\tau,C)) = (c\tau+d)^{\frac{h-1}{2}}(c\bar\tau+d)^{\frac{1}{2}}\,E[\frac{c}{c\tau+d}\frac{C^2}{2}]\,M(A)_{\gamma\delta}\,\Psi_\delta(\tau,C) \tag{2.57}$$

$$H_\gamma(A\cdot\tau) = \omega_A\,(c\tau+d)^{w_H}\,M(A)^{-1}_{\delta\gamma}\,H_\delta(\tau) \tag{2.58}$$

where $\omega_A^{-1}M(A)$ is a representation of $SL(2,\mathbb{Z})$ generated by

$$M(T)_{\gamma\delta} = \delta_{\gamma,\delta}\,e^{-i\pi(\gamma^{\|}+\frac{P}{2})^2} \tag{2.59}$$

$$M(S)_{\gamma\delta} = |\mathcal{D}|^{-1/2}e^{-2\pi i(\gamma^{\|}\cdot\delta^{\|}+\frac{P^3}{4})}e^{-i\frac{\pi}{4}(h-2)} \tag{2.60}$$

The phases ω_T,ω_S are given in (2.33), (2.36) above. It is worth noting that it is crucial to have the extra phase $e^{i\pi P\cdot F}$ in the partition function in order for the vector of functions Ψ_γ to transform into themselves.

[19] We noted in section 2.3 that $w=-3/2$ for proper Calabi-Yau manifolds but the following works for any value of w, so we will leave this an arbitrary parameter for now.

Now we would like to write a Poincaré series for H_γ. Since the modular weight $w_H = w - \frac{h-1}{2} = -1 - h/2$ is negative we should in fact first define the "dual" modular vector

$$\widetilde{H}_\gamma(\tau) := L^{1-w_H} H_\gamma(\tau), \qquad \text{where } Lf(\tau) := \frac{1}{2\pi i}\frac{\partial}{\partial\tau} f(\tau), \tag{2.61}$$

which transforms according to (2.58) but with modular weight $w_H \to 2 - w_H > 2$. The reason for this is the following nontrivial identity, which can be verified by elementary means and holds for any differentiable function f:

$$L^n \left[(c\tau+d)^{-1+n} f\left(\frac{a\tau+b}{c\tau+d}\right)\right] = (c\tau+d)^{-1-n}(L^n f)\left(\frac{a\tau+b}{c\tau+d}\right). \tag{2.62}$$

Next, it is convenient to define $j(A,\tau) := c\tau + d$ so that

$$j(A_1 A_2, \tau) = j(A_1, A_2\tau) j(A_2, \tau). \tag{2.63}$$

Finally, let Γ_∞ be the subgroup of Γ generated by $\tau \to \tau + 1$. Then we claim

$$\widetilde{H}_\gamma(\tau) = \sum_{A\in\Gamma_\infty\backslash\Gamma} (j(A,\tau))^{w_H-2}\, \omega_A^{-1} M(A)_{\delta\gamma} \widetilde{H}_\delta^-(A\cdot\tau) \tag{2.64}$$

Here $H_\gamma^-(\tau)$ is the *polar part* of $H_\gamma(\tau)$, namely, the terms in the sum (2.41) with negative powers of $e^{2\pi i\tau}$.[20] Equivalently, because of (2.41), these are the terms with positive $\widehat{q}_0$. Note that there is a finite number of such terms. Their physical interpretation will be given in the next section. The quotient by Γ_∞ is necessary because the factor $\omega_A^{-1} M(A)_{\delta\gamma}$ (2.64) cancels the transformation law of $\widetilde{H}_\gamma^-(\tau)$ for any $A = \tau \to \tau + b$, $b \in \mathbb{Z}$. The proof of (2.64) proceeds by noting that it is in the orthogonal complement of cusp forms, since it is in the image of the operator (2.61), but then it is completely determined by its Poincaré series. See [26, 27] for more details.

Again using (2.62), one can formally pull out the L^{1-w_H} operation on the right hand side so, formally at least, we have for the original H_γ:

$$H_\gamma(\tau) = h_\gamma + \sum_{A\in\Gamma_\infty\backslash\Gamma} (j(A,\tau))^{-w_H}\, \omega_A^{-1} M(A)_{\delta\gamma} H_\delta^-(A\cdot\tau) \tag{2.65}$$

where $h_\gamma(\tau)$ is some function such that $L^{1-w_H} h_\gamma(\tau) = 0$, i.e., h_γ is a polynomial in τ of order at most $|w_H|$. (We assume here that $|w_H|$ is integral. The case where $|w_H|$ is half-integral is more complicated and we do not fully understand it.)

Since $w_H < 0$ the series (2.65) is in fact not convergent. We can regularize it as follows. Using $A\tau = \frac{a}{c} - \frac{1}{c(c\tau+d)}$ we define the notation:

$$\left[e^{2\pi i k A\tau}\right]_N := e^{2\pi i k\frac{a}{c}} \left(e^{-2\pi i k \frac{1}{c(c\tau+d)}} - \sum_{j=0}^{N} \frac{1}{j!}\left(\frac{-2\pi i k}{c(c\tau+d)}\right)^j\right) \tag{2.66}$$

[20]Note that L commutes with taking the polar part, so the notation $\widetilde{H}^-$ is unambiguous.

and then, writing

$$H_\gamma(\tau) := \sum_k \hat{H}_\gamma(k) e^{2\pi i k\tau} \tag{2.67}$$

where k runs over $\frac{1}{M}\mathbb{Z}$ for some integer M, we replace the formal expression (2.65) by

$$H_\gamma(\tau) = h_\gamma + \sum_{A\in\Gamma_\infty\backslash\Gamma} (j(A,\tau))^{-w_H}\, \omega_A^{-1} M(A)_{\delta\gamma} \sum_{k<0} \hat{H}_\delta(k) \left[e^{2\pi i k A\tau}\right]_{|w_H|} \tag{2.68}$$

where h_γ is a polynomial of order $|w_H|$. We claim (2.68) transforms like a form of weight w_H, and extracting the degeneracies from the contour integral proceeds as in the case where we use the formal expression (2.65).

The Poincaré series representation of $H_\gamma(\tau)$ can be lifted to a Poincaré series representation of the partition sum $\mathcal{Z}(\tau, C)$ itself. Define the polar part of $\mathcal{Z}$ as

$$\mathcal{Z}^-(\tau, C) := \sum_\gamma \Psi_\gamma(\tau, C)\, H_\gamma^-(\tau). \tag{2.69}$$

Equivalently, this is $\mathcal{Z}$ truncated to the terms for which $\widehat{q}_0 > 0$. We can now substitute (2.68) into (2.39) and use (2.57) to get a Poincaré series for $\mathcal{Z}$. Introducing the slash operator

$$f|^A_{\nu,\bar\nu}(\tau, C) := (j(A,\tau))^{-\nu}(j(A,\bar\tau))^{-\bar\nu}\, \omega_A^{-1} E[-\frac{c}{c\tau+d}\frac{C^2}{2}]\, f(A\cdot(\tau, C)), \tag{2.70}$$

on arbitrary function $f(\tau, C)$, where $E[\ldots]$ was defined in (2.28), this can be written as

$$\mathcal{Z} = \sum_{A\in\Gamma_\infty\backslash\Gamma} \mathcal{Z}^-|^A_{\nu,\bar\nu} \tag{2.71}$$

where $\nu = w = -3/2$ and $\bar\nu = \bar w = 1/2$, and we dropped the divergent $\widehat{q}_0 = 0$ "counterterms" lifted from the h_γ, which are not important for the purpose of extracting $\widehat{q}_0 \neq 0$ degeneracies.

While (2.71) will be our main formula it is perhaps worth remarking that one could define a convergent Poincaré series for the quantity $\widetilde{\mathcal{Z}}$ analogous to $\widetilde{H}_\gamma$. To define $\widetilde{\mathcal{Z}}$, let us extend L as

$$L_- := \frac{1}{2\pi i}\frac{\partial}{\partial\tau} - \frac{1}{8\pi^2}\frac{\partial^2}{\partial C_-^2} \tag{2.72}$$

$$L_+ := \frac{1}{2\pi i}\frac{\partial}{\partial\bar\tau} - \frac{1}{8\pi^2}\frac{\partial^2}{\partial C_+^2} \tag{2.73}$$

$$L := L_- + L_+ = \frac{1}{2\pi i}\frac{\partial}{\partial C_0} - \frac{1}{8\pi^2}\frac{\partial^2}{\partial C^2}. \tag{2.74}$$

Plainly, $L_\pm$ annihilate Ψ_γ and hence

$$\widetilde{\mathcal{Z}} := L^{1-w_H}\mathcal{Z} = \sum_\gamma \Psi_\gamma(\tau, C)\widetilde{H}_\gamma(\tau). \tag{2.75}$$

Repeating the same steps as before, but now using (2.64), we get

$$\widetilde{\mathcal{Z}} = \sum_{A\in\Gamma_\infty\backslash\Gamma} \widetilde{\mathcal{Z}}^-|^A_{\nu,\bar\nu} \tag{2.76}$$

with $\nu = 2 - w_H + \frac{h-1}{2} = -w + h + 1 = h + 5/2$ and $\bar\nu = 1/2$.

3. BPS bound states in supergravity

3.1 Basic idea

In this section we will argue that the BPS states corresponding to the polar part of the partition function, i.e. D4-D2-D0 states with $\widehat{q}_0 > 0$, can be concretely thought of as bound states of D6 and anti-D6 branes (each with lower degree charges turned on), which moreover can be made to split into those two constituents by moving the background moduli t_∞ to a wall of marginal stability, implying in particular that the degeneracies of these states factorize accordingly. In a suitable asymptotic regime, this factorization of degeneracies translates in a factorization of the partition function roughly of the form

$$\mathcal{Z} \sim \mathcal{Z}_{\rm top}\, \overline{\mathcal{Z}_{\rm top}}, \tag{3.1}$$

in other words, to the OSV conjecture.

The starting point to derive this factorization is the observation, detailed below, that the polar charges do not have single centered black hole realizations in four dimensions, but instead are realized as two (or more) centered "molecular" bound states with nonparallel charges at the centers. This structure is mirrored to a certain extent in the microscopic D-brane description of these states, which we will develop in section 4.

Even without getting into any of the detailed descriptions, there is a simple physical argument for why polar states always "split," in the sense that they can be made to decay in constituents at some wall of marginal stability. This goes as follows.

The holomorphic central charge of the D4-D2-D0 system in the large radius approximation is given by (2.11):

$$Z = -\frac{1}{2} P^A D_{ABC} (B+iJ)^B (B+iJ)^C + Q_A (B+iJ)^A - q_0. \tag{3.2}$$

We claim that this has a zero in the interior of moduli space if and only if $\widehat{q}_0 := q_0 - D^{AB} Q_A Q_B > 0$, where we recall that $D_{AB} := D_{ABC} P^C$, $D^{AB} D_{BC} := \delta^A_C$. To see this, first make the change of variables $B \to \tilde{B}$:

$$B = \tilde{B} + D^{AB} Q_B. \tag{3.3}$$

Then

$$Z = -\frac{1}{2} D_{AB} (\tilde{B} + iJ)^A (\tilde{B} + iJ)^B - \widehat{q}_0. \tag{3.4}$$

Requiring $Z = 0$, leads to

$$\tilde{B} \cdot J = 0, \qquad \frac{1}{2}(J^2 - \tilde{B}^2) = \widehat{q}_0, \tag{3.5}$$

where as before the dot product is defined using the metric D_{AB}. Recall that J has positive norm squared and all vectors (in L_X) perpendicular to J have negative norm squared. Because of the first equation, $\tilde{B}$ is of this kind, hence the left hand side of the second equation is strictly positive in the interior of moduli space, so we need $\widehat{q}_0 > 0$. Conversely, when $\widehat{q}_0 > 0$, we can take for example $\tilde{B} = 0$, $J_0 = \sqrt{2\widehat{q}_0 / P^3}\, P$ and obtain $Z = 0$. This proves our claim. Note that at large P, this result is guaranteed to be robust under adding instanton corrections as long as $J_0^A \gg 1$. In particular this is true for the "most polar" states, that is states with $\widehat{q}_0$ near $(P^3 + c_2 P)/24$, which will be of main interest in the derivation of the OSV conjecture.

Now when the background moduli are at the zero locus at sufficiently large J, a BPS state of the given charge $\Gamma = (p, q)$ cannot exist; if it did, the charge would correspond to a massless BPS particle at this locus, which would cause a singularity of the moduli space metric [56, 57, 59]. Such singularities exist at conifold points of the mirror complex structure moduli space, but are (more or less by definition) absent in the large J region. Since by assumption the state does exist when $J \to \infty$, there must be a wall of marginal stability separating the zero locus from $J = \infty$. When crossing this wall of marginal stability coming from $J = \infty$, the state decays in two BPS states with charges Γ_1 and Γ_2, $\Gamma = \Gamma_1 + \Gamma_2$, whose central charges are aligned on the wall: $\alpha_1 \equiv \arg Z(\Gamma_1) = \alpha_2 \equiv \arg Z(\Gamma_2) = \alpha \equiv \arg Z(\Gamma)$.[21]

As we will review below, decay at marginal stability is realized in the supergravity picture by two (clusters of) centers moving infinitely far away from each other. In this infinite separation limit, one physically expects the degeneracies to factorize. In particular the Witten index Ω of this configuration, which is independent of the background moduli as long as the wall of marginal stability is not crossed, can be expected to have a factorized form $\Omega(\Gamma) = \Omega(\Gamma_2)\,\Omega(\Gamma_2)$. There is a slight subtlety however, in that quantizing the position degrees of freedom of the two parts produces an additional lowest Landau level degeneracy $|\langle\Gamma_1, \Gamma_2\rangle|$, where $\langle\Gamma_1, \Gamma_2\rangle$ is the Dirac-Schwinger-Zwanziger symplectic intersection product (A.2) on charge space. As we review under (3.24), this is most easily understood by noting that a 2-centered BPS bound state carries an intrinsic angular momentum $J = \frac{1}{2}(|\langle\Gamma_1, \Gamma_2\rangle| - 1)$, leading to an additional degeneracy $2J + 1$. Moreover, this intrinsic spin changes the fermion parity by a factor $(-1)^{2J} = (-1)^{\langle\Gamma_1,\Gamma_2\rangle - 1}$, which appears in the index Ω, defined as in (1.6)-(1.7). Thus, summing over different possible splits of Γ, we arrive at a factorization formula for polar states of the form

$$\Omega(\Gamma) = \sum_{\Gamma_1, \Gamma_2} (-1)^{\langle\Gamma_1,\Gamma_2\rangle - 1}\, |\langle\Gamma_1, \Gamma_2\rangle|\, \Omega(\Gamma_1)\, \Omega(\Gamma_2) \tag{3.6}$$

where the sum runs over allowed charge splits $(p, q) \equiv \Gamma = \Gamma_1 + \Gamma_2$ (with Γ_1, Γ_2 primitive). We have been sloppy here in the sense that we did not specify at which t the indices should be evaluated. We will make this precise in section 5.1.

Precisely which splits can be realized by decays of actual bound states and therefore have to be summed over is a highly nontrivial question, and analyzing this as well as to what extent it leads to the factorization (3.1) will in fact take up much of the remainder of this paper. As a byproduct of this analysis however, we will obtain several new insights in the structure of BPS states which are of independent interest.

[21] Recall that because of the BPS condition, decay is only energetically possible when the phases of the constituents align.

3.2 Review of BPS black hole bound states and attractor flow trees

In many cases, BPS D-brane states at $g_s|\Gamma| \ll 1$ correspond to single centered black holes in four dimensional supergravity at $g_s|\Gamma| \gg 1$. However, this is not always the case [58, 21]. It might even happen that a single centered BPS solution of the given charge does not exist at all. This is the case when the attractor flow corresponding to this charge terminates on a zero of the central charge Z at a regular point in moduli space [59, 60]. In such cases, it is necessary to consider more general multicentered BPS black hole bound states [21, 22, 23, 24]. These are stationary but in general non-static BPS solutions of supergravity [61, 62], with *non*-parallel charges at the centers. The distances between the centers are constrained by equations depending on the charges and the moduli at spatial infinity, and there is a potential energy exceeding the BPS bound when going off the constraint locus. Hence unlike the usual parallel charge multicentered BPS solutions, these are genuine bound states. Moreover, although time independent, they carry an intrinsic, quantized angular momentum, due to the Poynting vector field produced by the simultaneous presence of electric and magnetic charges.

As we saw above, the BPS states corresponding to the polar part of $\mathcal{Z}$ have regular zeros and therefore are of this type: they do not have single centered solutions, so they must have realizations as multicentered bound states. On the other hand D4-D2-D0 states with $\widehat{q}_0 < 0$ do have single centered solutions, but here we will find a surprise (described in section 3.5): when all charges are linearly scaled up by some sufficiently large Λ, in addition to the usual single centered solutions, there are always two-centered BPS configurations whose Bekenstein-Hawking entropy is parametrically larger than that of the single centered solution, growing as Λ^3 instead of the single centered growth Λ^2. Clearly, this creates some tension with the OSV conjecture, which predicts to leading order the single centered entropy growth $\log \Omega(\Lambda\Gamma) \sim \Lambda^2$. To what extent this is a problem for the conjecture will be discussed in detail in section 7.

3.2.1 General stationary BPS solutions

Let us now review the description of these solutions in more detail. The metric of a BPS solution is always of the form

$$ds^2 = -e^{2U}(dt+\omega)^2 + e^{-2U}d\vec{x}^2 \tag{3.7}$$

satisfying the BPS equations of motion[22]:

$$2\,e^{-U}\,\mathrm{Im}(e^{-i\alpha}\Omega_{\mathrm{nrm}}) = -H \tag{3.8}$$

$$*_3\,d\omega = \langle dH, H\rangle. \tag{3.9}$$

where $*_3$ is the Hodge star on flat $\mathbb{R}^3$, Ω_{nrm} and $\langle\cdot,\cdot\rangle$ are defined in appendix A, and $e^{i\alpha}$ is the phase of $Z(\Gamma;t)$. The function $H:\mathbb{R}^3 \to H^{\mathrm{even}}(X,\mathbb{R})$ is harmonic with poles at the centers. For an n-centered configuration with charges Γ_i in asymptotically flat space:

$$H(\vec{x}) = \sum_i \frac{\Gamma_i}{|\vec{x}-\vec{x}_i|} - 2\,\mathrm{Im}(e^{-i\alpha}\Omega_{\mathrm{nrm}})|_{r=\infty}. \tag{3.10}$$

[22]We neglect R^2 corrections [62], which is justified in the large charge limit

The phase field $\alpha(\vec{x})$ satisfies the boundary condition $\alpha|_{r=\infty} = \arg Z(\Gamma)|_{r=\infty}$, with Z given by (A.8).

In this subsection the period vector and central charge will always be the normalized versions, so to avoid cluttering the formulae we will henceforth not explicitly write the subscripts indicating this.

For a single center (3.8) reduces to the attractor flow equation

$$2\, e^{-U}\, \mathrm{Im}(e^{-i\alpha}\Omega) = -\Gamma\, \tau + \text{const.}, \qquad \tau \equiv 1/r. \tag{3.11}$$

The moduli at the horizon $\tau = \infty$ are fixed by the attractor equation

$$2\, \mathrm{Im}(\overline{Z(\Gamma; t_*(\Gamma))}\Omega) = -\Gamma, \tag{3.12}$$

and the Bekenstein-Hawking entropy is given by

$$S(\Gamma) = \pi |Z(\Gamma; t_*(\Gamma))|^2 \tag{3.13}$$

evaluated at the attractor point. Attractor flows are gradient flows of $\log|Z|^2$ [59, 60], hence the right hand side of this expression is minimized at the attractor point [63]. Equations (3.12)-(3.13) hold in the multicentered case for each center separately; in particular the attractor point and horizon area for each constituent black hole is not affected by the presence of the other centers.

Under the substitutions (1.10) and identifying $\Gamma = (p, q)$, the attractor equations can alternatively be written as [10]

$$2\pi q_\Lambda = \frac{\partial}{\partial \phi^\Lambda} \mathcal{F}_0(p, \phi). \tag{3.14}$$

where $\mathcal{F}_0 = \log|\mathcal{Z}_{\rm top}^{(h=0)}|^2$ is the genus zero[23] free energy [10], again with the substitutions (1.10). The entropy is then obtained as the Legendre transform of the free energy:

$$S(p, q) = \mathcal{F}_0(p, \phi) - 2\pi q_\Lambda \phi^\Lambda. \tag{3.15}$$

It was shown in [24] that (3.8)-(3.9) (as well as the equations giving the electromagnetic field) can be solved completely explicitly given just this entropy function. For example,

$$e^{-2U(\vec{x})} = S(H(\vec{x}))/\pi, \tag{3.16}$$

while the moduli fields $t^A(\vec{x})$ in our conventions are obtained as

$$t^A(\vec{x}) = \left. \frac{\frac{\partial S}{\partial q_A} + \pi i p^A}{\frac{\partial S}{\partial q_0} - \pi i p^0} \right|_{(p,q)=H(\vec{x})}. \tag{3.17}$$

Depending on the model and the charges, (approximate) expressions for S may or may not be obtainable analytically. In the large radius approximation, the general attractor solution

[23]The restriction to genus zero is due to the fact that we are neglecting R^2 corrections.

was derived in [64] (see also [59], sec. 9). Parametrizing a general charge $\Gamma \in H^{\rm even}(X,\mathbb{R})$ as

$$\Gamma = r e^S (1 - \beta + n\,\omega) \tag{3.18}$$

where $r \in \mathbb{R}$, $S \in H^2(X,\mathbb{R})$, $\beta \in H^4(X,\mathbb{R})$, $n\,\omega \in H^6(X,\mathbb{R})$ (with $\int_X \omega \equiv 1$), the condition to have an attractor point is:

$$\mathcal{D} := 8(Y^3)^2 - 9\,n^2 \geq 0, \quad Y^2 := \beta, \quad Y \in \text{ Kähler cone}, \tag{3.19}$$

with entropy

$$S = \frac{\pi}{3} r^2 \sqrt{\mathcal{D}}. \tag{3.20}$$

In this case, the region in $H^3(X,\mathbb{R})$ for which S is real and positive is the region for which the discriminant $\mathcal{D}$ is positive; we denote this region in general by $\mathrm{dom}\, S$.

Returning to the general multicentered case, we note that equation (3.9) has nonsingular solutions if and only if the following integrability condition, obtained by acting with $d{*_3}$ on both sides, is satisfied for all centers i:

$$\sum_{j=1(\neq i)}^{N} \frac{\langle \Gamma_i, \Gamma_j \rangle}{|\vec{x}_i - \vec{x}_j|} = 2\,\mathrm{Im}\left(e^{-i\alpha} Z(\Gamma_i)\right)_\infty . \tag{3.21}$$

In the case of just two charges Γ_1 and Γ_2, this simplifies to

$$|\vec{x}_1 - \vec{x}_2| = \frac{\langle \Gamma_1, \Gamma_2 \rangle}{2\,\mathrm{Im}(e^{-i\alpha} Z_1)_\infty} = \frac{\langle \Gamma_1, \Gamma_2 \rangle}{2} \left. \frac{|Z_1 + Z_2|}{\mathrm{Im}(Z_1 \bar{Z}_2)} \right|_\infty . \tag{3.22}$$

Since distances are positive, a necessary condition for existence in this case is

$$\langle \Gamma_1, \Gamma_2 \rangle \mathrm{Im}(Z_1 \bar{Z}_2)_\infty > 0. \tag{3.23}$$

From (3.22) it follows that the separation of the centers diverges when such a wall is approached from the side where the above inequality is satisfied. Thus, this process is the 4d supergravity realization of decay at marginal stability.

A crucial property of these multicentered solutions is that despite being time-independent, they carry intrinsic angular momentum [21], stored in the electromagnetic field, much as for electron-monopole pairs. This is given by

$$\vec{J} = \sum_{i<j} \frac{1}{2} \langle \Gamma_i, \Gamma_j \rangle \, \frac{\vec{x}_i - \vec{x}_j}{|\vec{x}_i - \vec{x}_j|}. \tag{3.24}$$

In particular for a two centered configuration, the angular momentum stored in the electromagnetic field equals $J = \frac{1}{2}|\langle \Gamma_1, \Gamma_2 \rangle|$. The presence of this angular momentum implies that quantizing this 2-particle "monopole-electron" system will give rise to a ground state degeneracy. The quantization of this system was studied in great detail in [65], also for more complicated multiparticle systems. One subtlety that was uncovered there was that the position hypermultiplet degrees of freedom of the particles arrange themselves such that the effective total angular momentum of the BPS ground state is lowered by 1/2, to a total of $J = \frac{1}{2}(|\langle\Gamma_1, \Gamma_2\rangle| - 1)$. This was derived explicitly in [65] by constructing the ground state wave functions, but if we think of the system as a light electron Γ_1 moving in the background field of a heavy monopole Γ_2, this can physically be understood as follows. If the electron Γ_1 lived in empty space, it would have a half-hypermultiplet $(\mathbf{0}, \mathbf{0}, \mathbf{\frac{1}{2}})$ of BPS states associated to its position degrees of freedom in $\mathbb{R}^3$. However in the case at hand it is moving in the magnetic field of the monopole Γ_2, and the interaction between this radial magnetic field and the hypermultiplet spin degrees of freedom in fact selects out a single energy minimizing state in the hypermultiplet, essentially spin 1/2 down in the radial direction. As a result, the total angular momentum is lowered by 1/2, as claimed, and the total ground state degeneracy (factoring out the decoupled center of mass half-hyper) equals $2J + 1 = |\langle\Gamma_1, \Gamma_2\rangle|$. This can also be interpreted as the lowest Landau level degeneracy of an electron confined on a sphere surrounding a magnetic monopole.

3.2.2 Existence criteria and attractor flow trees

Whether or not multicentered BPS solutions of given charges Γ_i and positions $\vec{x}_i$ actually exist is in general a rather nontrivial problem. Necessary and sufficient conditions are:

1. The integrability conditions (3.21).

2. To keep the metric warp factor real in (3.16), $H(\vec{x})$ must have positive discriminant $\mathcal{D}(H(\vec{x})) > 0$, that is, must lie in $\operatorname{dom} S$ for all $\vec{x} \in \mathbb{R}^3$.

3. The fields $t^A(\vec{x})$ must remain within the physical moduli space for all $\vec{x} \in \mathbb{R}^3$.

In particular, conditions such as $\Gamma_i \in \operatorname{dom} S$ for all i, and (3.21) are necessary but, in general not sufficient for existence of a BPS solution.

Now ideally, one would like to have a local necessary and sufficient existence criterion in terms of the charge and the background moduli only. The first condition above is local and easy to evaluate, but not sufficient, and the second and third conditions are not local, as they require information about fields at all $\vec{x}$. Unfortunately, a local necessary and sufficient existence criterion is not known, and given the intrinsic mathematical complexity of stability conditions in the theory of derived categories (see e.g. [66, 67, 68]), this is probably too much to hope for.

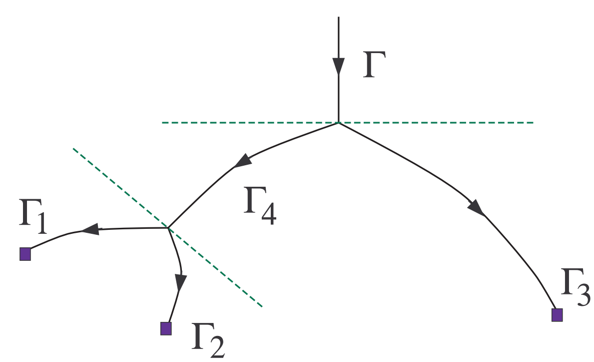


Figure 1: Sketch of an attractor flow tree. The dotted lines are lines of marginal stability and the squares are attractor points.

In [21], an existence criterion was proposed in terms of *attractor flow trees*, also called *split attractor flows*: a solution exists iff an attractor flow tree exists in moduli space starting at the background value of the moduli and terminating at the Γ_i attractor points.

Each edge E of an attractor flow tree is given by a single charge attractor flow for some charge Γ_E, charge and energy is conserved at the vertices, i.e. for each vertex $E \to (E_1, E_2)$, $\Gamma_E = \Gamma_{E_1} + \Gamma_{E_2}$ and $|Z(\Gamma_E)| = |Z(\Gamma_{E_1})| + |Z(\Gamma_{E_2})|$. The last condition is equivalent to requiring the vertices to lie on a line of marginal stability: $\arg Z(\Gamma_{E_1}) = \arg Z(\Gamma_{E_2})$. A number of arguments in favor of the equivalence with the full existence problem were given in [22], and a practical approach for computing split flows on the quintic was developed in [23].

The split flow approach gives a reasonably practical criterion in sufficiently simple examples, but it often requires case by case analysis, and is therefore perhaps not as powerful as one would wish as a general systematic test. Nevertheless, it will be quite useful in our analysis below. Note that it is not always necessary to construct the precise flow tree to argue for its existence; for example to argue for existence of a split flow with two endpoints, it is sufficient to establish the existence of the two attractor points and the existence of a wall of marginal stability between the starting point and the endpoint of the single flow with charge Γ (either a zero of Z or an attractor point).

In appendix D we outline an efficient algorithm for numerically checking existence of flow trees.

The general uplift of arbitrary multicentered IIA/CY$_3$ solutions to M-theory has been discussed in [70, 71, 72, 73], generalizing [74, 75, 76, 77, 78, 79] and relating some of these solutions in four dimensions to the multi-black hole/black ring and "bubbling" solutions in five dimensions studied e.g. in [80, 78, 79]; see [81] for a recent review. In [76, 78, 79, 73] it was pointed out that the 4d condition of having positive discriminant for $H(\vec{x})$ everywhere is equivalent to the 5d condition of having no closed timelike curves, which is similarly nontrivial to verify directly. Through the correspondence given in those works, the flow tree picture reviewed here is thus directly applicable to existence and classification of 5d solutions as well.

In any case, what emerges from examples is that existence of a certain $\Gamma \to \sum_i \Gamma_i$ bound state realization is highly constrained; in particular, although a priori there is an infinite number of ways of splitting up a given charge, only a finite number of those turn out to correspond to a flow tree. Physically this is as expected, since an infinite number would imply an infinite degeneracy of BPS states of a given charge. [24] Some more direct general arguments, based on the monotonic decrease of $|Z|$ along attractor flows, were given in appendix A of [23]. Part of this argument is made more precise in appendix C.

To summarize: Throughout this paper we will assume the truth of the following *split attractor flow conjecture*, which we consider to be very well-founded:

Split Attractor Flow Conjecture:

a) The components of the moduli spaces (in $\vec{x}_i$) of the multicentered BPS solutions with constituent charges Γ_i and background t_∞, are in 1-1 correspondence with the attractor flow trees beginning at t_∞ and terminating on attractor points for Γ_i.

[24] Even if some states associated with different flow trees mixed and were lifted quantum mechanically to near-BPS states, the number of states below any finite energy scale should be finite.

b) For a fixed t_∞ and total charge Γ there are only a finite number of attractor flow trees.

We note the following subtleties:

- Finiteness is of course only valid when charge quantization is imposed, and hence is not strictly speaking a property of the classical theory.

- It is useful to distinguish between attractor flows terminating on regular points in moduli space and those associated to pure electric or pure magnetic charges which flow off to infinity. Since the latter case leads to (mildly) singular solutions, some argument that transcends supergravity is strictly speaking required to establish the existence of the corresponding BPS states.

- It is *not* true that a single flow only corresponds to a single centered solution. Indeed, as we will see in section 3.8, there exist multicentered solutions which are in some sense continuously connected to a single centered solution. These are the so-called "scaling solutions", first identified in [65], which develop a capped off $\mathrm{AdS}_2 \times S^2$ throat with a scale modulus λ parametrizing the depth of the throat, and which asymptotically for $\lambda \to 0$ become indistinguishable from a single centered black hole for a distant observer. Such configurations cannot be forced to decay at a wall of marginal stability, and thus are not described by a split attractor flow (since a split flow can always be made to decay by crossing the wall of marginal stability on which the split occurs).

3.2.3 Attractor flow trees and the Hilbert space of quantum BPS states

So far we discussed the relation between attractor flow trees and classical BPS solutions of four dimensional supergravity. However, the attractor flow criterion for existence of BPS states can be argued to be valid beyond the classical four dimensional supergravity picture. As we mentioned above, even for purely electric charges, which lead to singular 4d gravity solutions and are better described as probe particles, the flow tree picture continues to hold. Moreover, flow trees can be given a purely microscopic interpretation. This was done for the IIB description of BPS D-brane as special Lagrangians in [69], and we will sketch a more general argument based on tachyon condensation of open stretched strings in section 4.1. Finally, after quantization of the four dimensional BPS configuration moduli space of a given charge Γ with moduli at infinity t_∞, the partitioning of this moduli space by attractor flow trees leads to a partitioning of the corresponding BPS Hilbert space $\mathcal{H}(\Gamma; t_\infty)$. This suggests that attractor flow trees provide a classification of BPS states independent of any particular picture, more refined than classification by total charge only, but coarser than distinguishing individual states.

In fact a general physical argument for this idea can be given, as follows. The starting point is the physical expectation that at an attractor point for charge Γ_i, (irreducible) BPS states exist with that charge, while at a zero of the central charge at a nonsingular point of the moduli space, there cannot be any BPS states (since zero mass BPS states lead to singularities). The crucial second ingredient is the observation that if a BPS state of some charge Γ exists at a certain point in moduli space, it will continue to exist at all points along the attractor flow for that charge Γ passing through this point, when one follows the flow in the *inverse* direction, that is *decreasing* τ in (3.11). This can be seen as follows. A BPS state can only disappear when it decays at a wall of marginal stability, when crossing the wall from the side where the stability condition (3.23) is satisfied to the side where it is not. However, an inverted attractor flow will always cross any such wall in the opposite direction, i.e. from unstable to stable. Indeed, say we are near a wall of $\Gamma \to \Gamma_1 + \Gamma_2$ marginal stability. By taking the intersection product of (3.11) with Γ_1, it follows that $2\, e^{-U} \mathrm{Im}(e^{-i\alpha} Z_1) = -\langle \Gamma_1, \Gamma \rangle\, \tau + \text{const.}$, which can also be written as

$$2\, e^{-U}\, \mathrm{Im}(Z_1\, \bar{Z}_2) = -\langle \Gamma_1, \Gamma_2 \rangle\, |Z| \tau + \text{const.} \tag{3.25}$$

From this it is clear that $\langle \Gamma_1, \Gamma_2 \rangle \mathrm{Im}(Z_1\, \bar{Z}_2)$ can only increase along an inverted attractor flow, which proves our claim. Thus, if we have a split attractor tree, we can start with BPS states of charge Γ_i at the attractor points, let them flow up along the tree edges, "glue" them together (microscopically through tachyon condensation as will be reviewed in section 4.1, macroscopically by creating multicentered configurations, initially with infinitely separated centers) at the MS vertices, as described in section 3.1, and then continue to flow up with this newly formed BPS state, all the way to the starting point of the tree, where we end up with a BPS state of the required total charge Γ.

Conversely, we can start with a charge Γ and some point t in moduli space, and consider the Hilbert space $\mathcal{H}(\Gamma; t)$ of BPS states with charge Γ at t. When flowing down along the attractor flow starting at t, some states might decay by splitting in two BPS states at walls of marginal stability, reducing the Hilbert space in size. Whenever such a decay occurs, we can associate to this event a flow split in the obvious way. The procedure can be repeated for each of the constituents separately starting from the split point, and so on, until each flow branch terminates in an attractor point. This algorithm decomposes $\mathcal{H}(\Gamma; t)$ in sectors labeled by different flow trees, according to the decay pattern under the procedure just described.

Thus we arrive at the picture that flow trees label different sectors of the Hilbert space of BPS states of a given charge in a given background, independent of the description of these states.

We note the following subtleties:

- Although every flow tree is associated to a component of the moduli space of classical BPS solutions, not all of these components survive quantization. The reason for this is the Pauli exclusion principle; for example, even if classically we can form a bound state of some charge Γ_1 with an arbitrary number of charges Γ_2, if the Γ_2 particles happen to be fermions and their number is larger than the number of available one-particle BPS ground states, the exclusion principle forbids a BPS bound state. This was discussed in detail in [65].

- The different sectors of $\mathcal{H}(\Gamma; t_\infty)$ labeled by different flow trees are not necessarily superselection sectors, as quantum tunneling might occur between different configurations with the same charge. For the same reason, part of the BPS states obtained say by quantizing different classical components of moduli space might in fact be lifted due to quantum tunneling. Presumably these tunneling amplitudes are exponentially small in some measure of the charges involved. Similarly, tunneling phenomena may occur when starting from the microscopic D-brane picture of these states at zero string coupling. In this case the suppression can be expected to be exponentially small in the inverse string coupling. However, the index is of course not affected by this, and can be computed in any semiclassical picture. It would be interesting to investigate these tunneling phenomena in more detail.

3.3 Symmetries

Scaling symmetries will be a powerful tool in the following, so we describe these in detail here.

The BPS equations of motion (neglecting R^2 corrections) always have the following scaling symmetry

$$\Gamma \to \mu\Gamma, \qquad t^A \to t^A, \qquad g_{\rm top} \to g_{\rm top}/\mu, \qquad \vec{x} \to \mu\vec{x}, \tag{3.26}$$

with $g_{\rm top}$ defined as in (1.10), while the OSV potentials scale as $\phi \to \mu\phi$. Under this scaling, the leading order entropy (i.e. without R^2 corrections governed by the higher genus topological string amplitudes) scales as $S \to \mu^2 S$. Moreover in the large μ limit, R^2 corrections can be consistently neglected, so this scaling becomes exact.

In the large radius regime, dropping all instanton corrections, there is a less trivial additional scaling symmetry:

$$(p^0, P, Q, q_0) \to (p^0, \lambda P, \lambda^2 Q, \lambda^3 q_0), \qquad t^A \to \lambda t^A, \qquad g_{\rm top} \to g_{\rm top}, \qquad \vec{x} \to \lambda^{3/2}\vec{x}, \tag{3.27}$$

with the OSV potentials remaining invariant. The corresponding leading order entropy scales as $S \to \lambda^3 S$. Moreover in the large λ limit, instanton corrections can be consistently neglected, so this scaling becomes exact.

There are also two useful discrete symmetries. The first one is simply charge conjugation $\Gamma \to -\Gamma$ with everything else invariant. This is valid in all regimes. The second is only valid in the large radius regime and given by

$$\Gamma \to \Gamma^*, \quad \text{i.e. } (p^0, P, Q, q_0) \to (p^0, -P, Q, -q_0), \qquad B \to -B. \tag{3.28}$$

This leaves the entropy invariant but inverts intersection products: $\langle \Gamma_1^*, \Gamma_2^* \rangle = -\langle \Gamma_1, \Gamma_2 \rangle$. Furthermore, the central charges transform according to

$$Z(\Gamma^*; t) = -\overline{Z(\Gamma; -\bar{t})}. \tag{3.29}$$

Microscopically it corresponds to taking the dual of the object in the derived category. In the case of objects described by vector bundles, this simply amounts to inverting the curvature, $F \to -F$.

Finally, there is a gauge symmetry

$$\Gamma \to e^{S}\Gamma, \qquad B \to B + S. \tag{3.30}$$

If we neglect charge quantization, this is a continuous symmetry, otherwise S has to be integral. This descends from the usual gauge symmetry which simultaneously shifts B and the worldvolume flux F. Note that *if there are no walls of marginal stability between $B+iJ$ and $B+S+iJ$*, then $B \to B+S$ with fixed Γ is a symmetry of the BPS spectrum, and hence because of the above gauge symmetry, likewise $\Gamma \to e^{S}\Gamma$ with fixed B is a symmetry of the BPS spectrum. This is the case for D4-D2-D0 systems in the large J limit. As we will see though, this is *not* so in general for D6-D4-D2-D0 systems, not even at $J \to \infty$.

3.4 A class of examples

We will now give a class of explicit examples relevant to our analysis below. Consider the charges

$$\begin{aligned}
\Gamma_1 &= r\, e^{\frac{P}{2r}}\,(1 - \tilde{\beta}\,\frac{P^2}{r^2} - \tilde{n}\,\frac{P^3}{r^3}) && (3.31)\\
&= r + \frac{P}{2} + (\frac{1}{8} - \tilde{\beta})\frac{P^2}{r} + (\frac{1}{48} - \frac{\tilde{\beta}}{2} - \tilde{n})\frac{P^3}{r^2} && (3.32)\\
\Gamma_2 &= -r\, e^{-\frac{P}{2r}}\,(1 - \tilde{\beta}\,\frac{P^2}{r^2} + \tilde{n}\,\frac{P^3}{r^3}) && (3.33)\\
&= -r + \frac{P}{2} - (\frac{1}{8} - \tilde{\beta})\frac{P^2}{r} + (\frac{1}{48} - \frac{\tilde{\beta}}{2} - \tilde{n})\frac{P^3}{r^2} && (3.34)\\
\Gamma &:= \Gamma_1 + \Gamma_2 = P + (\frac{1}{24} - \tilde{\beta} - 2\tilde{n})\frac{P^3}{r^2}. && (3.35)
\end{aligned}$$

Here $r > 0$ is a D6-charge, $P = P^A D_A \in H^2(X)$ a D4-charge which we take to be inside the Kähler cone (i.e. $P > 0$), the terms proportional to $P^2 = D_{ABC}P^A P^B \widetilde{D}^C \in H^4(X)$ are D2-charges and those proportional to P^3 are D0-charges.[25] We will work in the large radius and large charge approximation, i.e. we will retain only the cubic part of the prepotential.

We choose this parametrization such that $\tilde{\beta}$ and $\tilde{n}$ are invariant under the rescalings discussed in the previous subsection, and to simplify the entropy formulas of the two constituents as much as possible. Note that Γ_2 is the conjugate dual charge to Γ_1 (i.e. the Γ_2 is the image of Γ_1 under the combined action of the two discrete symmetries described in the previous subsection). This makes this class of examples particulary symmetric. The case $\tilde{\beta} = \tilde{n} = 0$ corresponds to the bound state of a pure D6 with flux $F = \frac{P}{2r}\,\mathbf{1}_r$ and the anti-brane of a pure D6 with flux $F = -\frac{P}{2r}\,\mathbf{1}_r$.[26]

In what follows it will also be convenient to use the variables

$$\nu := \frac{1}{24} - \tilde{\beta} - 2\tilde{n}, \qquad \mu := \frac{1}{8} - \tilde{\beta}, \tag{3.36}$$

which are proportional to the D0 resp. D2 charges of the constituents.

[25] We consider P^3 to be an element of $H^6(X)$ or a real number, depending on context.

[26] Since we work in the supergravity approximation in this subsection, we ignore flux quantization.

When the total D0-charge is negative, i.e. $\nu < 0$, there exists a regular attractor point for the total charge Γ, at

$$B = 0, \quad J = \sqrt{-6\nu}\,\frac{P}{r}, \quad g_{\rm top} = \frac{\pi\sqrt{-48\nu}}{r} \quad \text{with entropy } S = 2\pi\sqrt{-\nu/6}\,\frac{P^3}{r}. \tag{3.37}$$

On the other hand, as we saw before, when $\nu > 0$, $Z(\Gamma)$ has a zero locus; for example $Z(\Gamma) = 0$ at $B = 0$, $J = \sqrt{2\nu}\frac{P}{r}$. Therefore the attractor flow associated to Γ will crash on a regular zero, and no single centered BPS solution exists.

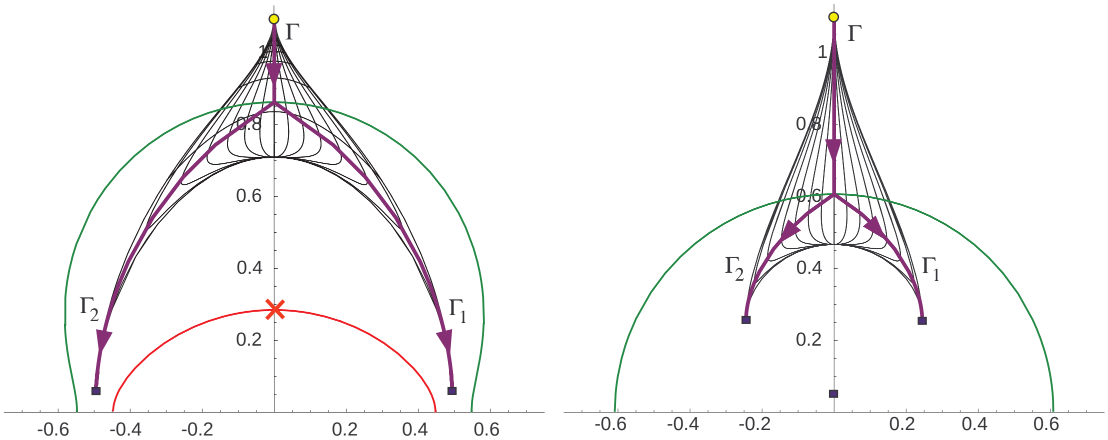


Figure 2: Left: Bound state features of the z-plane for parametrization $B+iJ = zP/r$, for a polar case $\tilde{\beta} = 1.25 \times 10^{-3}$, $\tilde{n} = 0$, $z_\infty = 1.1\,i$. The green (upper) line is the line of marginal stability, where the phases of Z_1 and Z_2 align. On the red (lower) line, the phases anti-align. The red cross is the zero of $Z(\Gamma)$. The fat split purple line is the attractor flow tree, and the black lines forming a pair of pants around this skeleton are the image of the moduli field $z(\mathbb{R}^3)$, following radial lines (and a few r = constant lines) out of the midpoint between the centers $\vec{x} = 0$. **Right:** Analogous plot for a nonpolar case $24\nu = -.01$, $8\mu \approx 0.49$ ($\tilde{\beta} \approx 0.064$, $\tilde{n} \approx -0.01$). The blue square on the imaginary axis is the attractor point of the single flow for Γ which exists for this value of ν.

Thus, to verify if Γ exists as a BPS bound state of Γ_1 and Γ_2 when $\nu > 0$, we first need to check if a wall of $\Gamma \to \Gamma_1 + \Gamma_2$ marginal stability exists between the zero and the value of the moduli at spatial infinity. Taking $B = 0$ at spatial infinity, we can follow a path from the moduli there to the zero locus parametrized by $B(y) = 0$, $J(y) = yP/r$, where y goes from $y = y_\infty$ (which we can think of as very large, since we are primarily interested in BPS states in the large radius limit) to $y = \sqrt{2\nu}$. Along this path

$$Z(\Gamma_1) = \left(-\frac{iy^3}{6} + \frac{y^2}{4} + i\mu y - \frac{\nu}{2}\right)\frac{P^3}{r^2}, \qquad Z(\Gamma_2) = \overline{Z(\Gamma_1)}. \tag{3.38}$$

Note that the phases of $Z(\Gamma_1)$ and $Z(\Gamma_2)$ align iff they are both real, which is the case at

$$y = y_{\rm ms} = \sqrt{6\mu}. \tag{3.39}$$

Thus for this to happen along the path when $y_\infty \to \infty$, we need

$$\mu \geq \frac{\nu}{3} \quad (\text{for } \nu \geq 0), \qquad \text{i.e.} \quad \tilde{\beta} - \tilde{n} \leq \frac{1}{6} \quad (\text{for } \tilde{\beta} + 2\tilde{n} \leq \tfrac{1}{24}). \tag{3.40}$$

In the nonpolar case $\nu < 0$, although there are single centered black hole solutions, there might still be 2-centered solutions as well. In other words it might happen that both a single flow and a split flow exists for a given charge. For this to happen, the wall of marginal stability must separate the attractor point from the value of the moduli at infinity. This leads to

$$\mu \geq -\nu \quad (\text{for } \nu \leq 0), \qquad \text{i.e.} \quad \tilde{\beta} + \tilde{n} \leq \frac{1}{12} \quad (\text{for } \tilde{\beta} + 2\tilde{n} \geq \tfrac{1}{24}). \tag{3.41}$$

It is instructive to evaluate the necessary condition for existence (3.23), which in the case at hand gives

$$\langle \Gamma_1, \Gamma_2 \rangle \, \mathrm{Im}(Z_1 \overline{Z_2})_\infty \sim (\mu + \nu)(y_\infty^2 - 2\nu)(y_\infty^2 - 6\mu) > 0. \tag{3.42}$$

Note that although this is always positive in the limit $y_\infty \to \infty$ when $\mu > -\nu$, this is not enough to guarantee the flow splits, as we just saw. Although the stable side of the marginal stability line $y = \sqrt{6\mu}$ *near* this line is characterized by a positive value of the left hand side, the latter becomes also positive when we continue into the unstable side and cross the line of *anti*-marginal stability $y = \sqrt{2\nu}$, where the phases *anti*-align. To guarantee that y_∞ does not lie in this region, we need that the marginal stability line lies above the anti-marginal stability line, i.e. $6\mu > 2\nu$.

Finally, note that when the background moduli are chosen to be at the attractor point for Γ, the stability condition (3.23) is not satisfied, so there will in any case be no 2-centered bound state for this value of the moduli. This is true in general, being a direct consequence of (3.25).

Of course, (3.40) or (3.41) are not sufficient to guarantee the existence of a 2-centered black hole solution based on the split flow with two endpoints. In addition we need both Γ_1 and Γ_2 to have attractor points, i.e. (3.19) needs to be satisfied. In the case at hand this reduces for both centers to

$$\mathcal{D} = 8\tilde{\beta}^3 - 9\tilde{n}^2 \geq 0. \tag{3.43}$$

The attractor point is then $B + iJ = z_*(\Gamma_i)\,\frac{P}{r}$ with

$$z_*(\Gamma_1) = \frac{1}{2} + \frac{3\tilde{n} + i\sqrt{\mathcal{D}}}{2\tilde{\beta}}, \qquad z_*(\Gamma_2) = -\frac{1}{2} + \frac{-3\tilde{n} + i\sqrt{\mathcal{D}}}{2\tilde{\beta}}. \tag{3.44}$$

The corresponding entropy is

$$S_1 = S_2 = \frac{\pi}{3}\sqrt{\mathcal{D}}\,\frac{P^3}{r}. \tag{3.45}$$

Note that when $r = 1$, the total charges in the limiting case $\tilde{\beta} = \tilde{n} = 0$ are exactly those of a pure D4-brane wrapped on P. In particular the D0 charge is $\widehat{q}_0 = q_0 = P^3/24$,

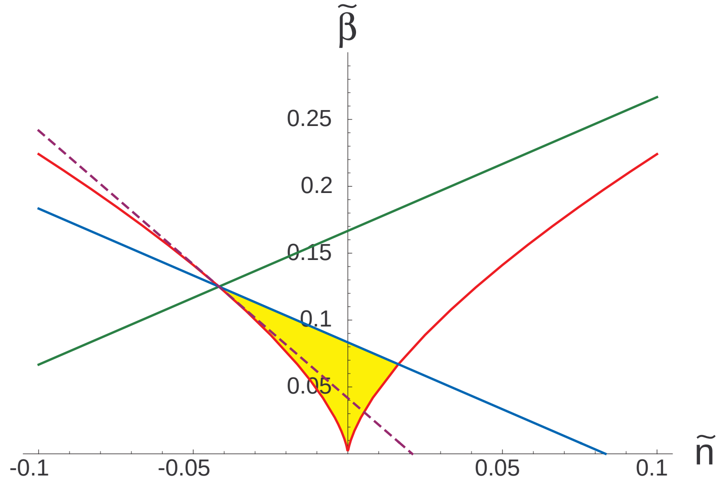


Figure 3: Region in $(\tilde{n}, \tilde{\beta})$-space supporting two-centered black hole bound states (yellow shaded triangle). The red curved line is the black hole bound (3.43), the upper green line represents the split bound (3.40) for the polar case and the lower blue line is the split bound (3.41) for the nonpolar case. The purple dotted line, corresponding to $\nu = 0$, separates polar from nonpolar charges; the polar region lies below. The blue boundary corresponds to the limit in which the intersection product and hence the separation between the two centers vanishes. The rightmost boundary vertex gives the most negative value of the total D0-charge ν, $\nu_{\rm min} = (3 - 2\sqrt{3})/8 \approx -0.058$. The origin gives the most positive value, $\nu_{\rm max} = 1/24$. The vertex on the left through which all lines pass has $(\tilde{n}, \tilde{\beta}) = (-1/24, 1/8)$, $(\mu, \nu) = (0, 0)$.

which as we saw in section 2, eq. (2.42) is indeed the highest possible value of $\widehat{q}_0$.[27] This suggests that the pure D4 is in fact a bound state of a pure $D6$ plus flux and a pure anti-D6 plus flux. We will confirm this picture in detail in the next section, both microscopically and macroscopically. The cases with $\tilde{\beta}, \tilde{n}$ small then correspond to adding a "dilute gas" of D2 and D0 branes to these D6 and anti-D6 branes.

For $r > 1$, the total D0-charge is not that of a single smooth D4-brane wrapping the class P, but rather that of r D4-branes each wrapping the class P/r, as might have been expected from a bound state of rank r D6 and anti-D6 branes. In particular the D0-charge is $r\,(P/r)^3/24$, which when P is very large has a large gap to the most polar charge $P^3/24$ obtained at $r = 1$. Hence such configurations enter far from the most polar terms in the partition sum $\mathcal{Z}$. This will be important for the derivation of the OSV conjecture later on.

3.5 The Entropy Enigma

The example of the previous section illustrates a remarkable phenomenon, namely, when we scale up a nonpolar total charge $\Gamma \to \Lambda\Gamma$, with $\Lambda \to \infty$, *the 2-centered BH entropy*

[27] at least in the large charge supergravity approximation in which we are working in this subsection, which drops the subleading $c_2 P/24$ correction. It is not hard to check that this correction is also correctly reproduced after taking into account the c_2 corrections to the Γ_i.

dominates over the single centered BH entropy, scaling as Λ^3 *as opposed to the single centered* Λ^2*!* Here we define the 2-centered Bekenstein-Hawking entropy as the sum of the Bekenstein-Hawking entropies of the two individual constituent black holes.

First, let us recall from section 3.3 that in the large Λ limit, the single centered entropy always scales as

$$S_{1\,\mathrm{center}}(\Lambda\Gamma) = \Lambda^2 S(\Gamma). \tag{3.46}$$

This is easily seen to be true for (3.37), but extends beyond the large radius approximation in which that expression is valid.

Now let us compare this to the two centered case. To illustrate our point, consider the case $r = 1$, $\tilde{n} = 0$, $\tilde{\beta} = 1/24$, which is clearly inside the stability domain of fig. 3, and corresponds to $\nu = 0$, $\mu = 1/12$. The total charge is then simply $\Gamma = P$, so we achieve uniform scaling by $P \to \Lambda P$. But then from (3.45)

$$S_{2\,\mathrm{centers}} = S_1 + S_2 \to \frac{\pi}{36\sqrt{3}}\Lambda^3 P^3. \tag{3.47}$$

Thus we get the claimed Λ^3 scaling. Note that this is not the only configuration with total charge $\Gamma = P$. Other configurations will have different numerical prefactors replacing $\frac{\pi}{36\sqrt{3}}$.

This behavior is completely generic and valid for *any* D4-D2-D0 charge Γ which is uniformly scaled up (if $P > 0$). To see this, first note that because of the shift symmetry discussed at the end of section 3.3, we can assume the D2-charge to be zero without loss of generality. Then we can as above split $\Gamma = (0, \Lambda P, 0, \Lambda q_0)$ into $\Gamma_{1,2} = \pm r + \Lambda\frac{P}{2} \pm \Lambda^2\mu\frac{P^2}{r} + \Lambda^3\frac{q_0}{2\Lambda^2}$ for some suitably chosen μ. Because of the second scaling symmetry discussed in section 3.3, this exists as a 2-centered solution iff the split into $\Gamma_{1,2} = \pm r + \frac{P}{2} \pm \mu\frac{P^2}{r} + \frac{q_0}{2\Lambda^2}$ exists. In the limit $\Lambda \to \infty$, the last term (the D0-charge) can be neglected, so this boils down to existence of a 2-centered configuration in the class of examples given above with $\nu = 0$. Obviously there are plenty of such configurations; the example given in the previous paragraph is one possibility, but it is easy to see that there is a whole family of more general choices of r, $\tilde{\beta}$ and $\tilde{n}$ leading to a 2-centered configuration.

For any such choice, the total BH entropy is nonzero and scales as Λ^3, because this is how S scales under the second scaling symmetry of section 3.3.

This establishes the existence of 2-centered BPS black hole bound states at J_∞ sufficiently large ($> \mathcal{O}(\Lambda)$) for *any* charge $\Lambda\Gamma$ where $\Gamma = (0, P > 0, Q, q_0)$ and $\Lambda \to \infty$, with entropy scaling as Λ^3 rather than the single centered scaling Λ^2. Note however that when J_∞ is kept fixed at some finite, Λ-independent value, eventually, the 2 centered solutions will cease to exist. The reason is that the wall of marginal stability for the configuration lies at a value of J of order Λ, which when $\Lambda \to \infty$ runs off to infinity, moving our background point out of the stability domain.

Note also that this is not in contradiction with the microscopic computation of the entropy of D4-D2-D0 systems in [2, 3] and its successful matching with the single centered entropy, since the regime of validity of this computation is $|\hat{q}_0| \gg P^3$, precisely the regime in which there are *no* multicentered solutions, and a regime from which one automatically exits when all charges are uniformly scaled up.

Nevertheless, since this Λ^3 scaling is surprising, to say the least, in the remainder of this section we will justify carefully the validity of these solutions, and of the entropy computed from them.

Let us fix a particular two centered solution and denote the fields and parameters associated to this by a subscript 0, e.g. Γ_0 is the total charge, $B_0 + iJ_0$ the moduli fields and so on. For simplicity, let us more concretely consider some case with $\nu = 0$ in our class of examples (so that $\Gamma_0 = P_0$ and scaling P_0 is equivalent to scaling Γ_0), in some asymptotic background $(B_0 + iJ_0)|_\infty = z_0|_\infty \frac{P}{r}$ with $z_0|_\infty$ above the line of marginal stability. We can scale up

$$r \to \xi\, r_0, \qquad P \to \xi\, \Lambda\, P_0 \tag{3.48}$$

without affecting the split attractor flow in rescaled coordinates z defined by $B + iJ =: z\frac{P}{r}$. Since $\mathrm{Im}\, z$ stays bounded away from zero, we thus see that when $\Lambda \to \infty$, we have $J = J_0 \Lambda \to \infty$ and the large CY radius approximation (dropping instanton corrections) is justified.

Note that the ξ-scaling implements the symmetry (3.26), while the Λ-scaling implements (3.27). Consequently all characteristic length scales L of the four dimensional solutions can be expressed in the form

$$L = c_0\, \xi\, \Lambda^{3/2}\, \ell_4, \tag{3.49}$$

where c_0 depends only on r_0, P_0, $\tilde{n}_0$, $\tilde{\beta}_0$ and $t_0|_\infty$. Hence all curvature radii in 4d Planck units go to infinity when $\Lambda \to \infty$. Note that this scaling also implies the Λ^3 scaling of the entropy is consistent with holography, since the area in Planck units of any surface enclosing the centers will scale as Λ^3.

To express L in string units, we use $\ell_4 = g_{4d}\ell_s$ where g_{4d} is the four dimensional IIA string coupling constant, related to the ten dimensional g_{IIA} by $g_{4d}^2 = g_{\mathrm{IIA}}^2/V_{\mathrm{IIA}}$, with $V_{IIA} = J^3/6$ the IIA CY volume in string units. Considered as a field, $g_{4d}(\vec{x})$ sits in a hypermultiplet and does not vary over space (so $g_{\mathrm{IIA}}(\vec{x})$ does vary, since $J(\vec{x})$ does). Hence, keeping the asymptotic value of g_{IIA} fixed at $g_{IIA,0}$, we have the scaling

$$L = \frac{c_0\, g_{\mathrm{IIA},0}}{\sqrt{V_{\mathrm{IIA},0}}}\, \xi\, \ell_s, \tag{3.50}$$

where $g_{\mathrm{IIA},0}$ and $V_{\mathrm{IIA},0}$ should be thought of as asymptotic values at spatial infinity. Note that equation (3.50) no longer scales with Λ, but we can still make it as large as we wish by scaling up ξ, i.e. by considering large r, so at least in this regime higher order curvature corrections are certainly under control, and we have no reason left to doubt our solutions. Related to this, note that the effective topological string coupling constant as given by (1.10), which controls F-term R^2 corrections, does not scale with Λ though it does scale as $1/\xi$, according to (3.26) and (3.27).

One could worry about cases with small r, since in this case at small $g_{\mathrm{IIA},0}$ the characteristic distance scales are small in string units, so one might fear that R^2 corrections will get out of control and we cannot trust the entropy formula we found. However in this case we can switch to the M-theory description using the 4d-5d correspondence of

[70, 71, 76, 77] to get a reliable picture. To achieve this, instead of keeping the asymptotic value of $g_{\rm IIA}$ fixed, we let it scale with Λ to keep the M-theory CY volume in 11d Planck units $V_M = g_{4d}^{-1}$ fixed, which amounts to taking $g_{\rm IIA}(\Lambda) = \Lambda^{3/2} g_{\rm IIA,0}$. Since $g_{\rm IIA} = R_M^{3/2}$ where R_M is the radius of the M-theory circle in 11d Planck units, this means we have

$$R_M = \Lambda R_{M,0}, \qquad L = \frac{c_0}{\sqrt{R_{M,0} V_{M,0}}}\, \xi\, \Lambda^{3/2}\, \ell_{11} \tag{3.51}$$

where we used $g_{\rm IIA,0}\ell_s = R_{M,0}\ell_{11}$. Hence we see that all characteristic length scales of the solution (including R_M) go to infinity in 11d Planck units when $\Lambda \to \infty$. Moreover the M-theory CY volume $V_M = V_{M,0} = R_{M,0}^{-3} J_0^3/6$ is constant over space, remains constant under the scalings and can be taken as large as we wish (as it is a hypermultiplet scalar). Finally we can also take $R_{M,0}$ as large as we wish, by taking the IIA asymptotic Kähler class J_0 large (although this changes a vector multiplet scalar, we saw this preserves the 2-centered solution).

The 4d solution near the D6D4D2D0 centers lifts up to a 5d BMPV spinning M2 black hole with $q_{M2} = \pm\tilde{\beta}\frac{P^2}{r} \sim \xi\Lambda^2$, $J_L^3 = \frac{1}{2}\tilde{n}\frac{P^3}{r^2} \sim \xi\Lambda^3$ located at the center of a $\mathbb{Z}_r$ quotient of Taub-NUT [70], which was shown in [70] to have exactly the same Bekenstein-Hawking entropy as the corresponding 4d black hole, scaling as $S \sim \xi^2\Lambda^3$ in the case at hand.

Thus we conclude that even for small r, the enigmatic Λ^3 entropy growth we find is reliable.

From the point of view of the topological string, this is perhaps more surprising. Solving (3.14) for the Γ_1 attractor point and substituting this in (1.10), we find

$$g_{\rm top} = \frac{4\pi}{\frac{-3r\tilde{n}}{\sqrt{\mathcal{D}}} + ir} \tag{3.52}$$

with $\mathcal{D}$ as in (3.43). This does not scale with Λ and is generically of order 1 for r of order 1. How can it be then that R^2 corrections to the entropy (obtained by replacing the genus zero $\mathcal{F}_0$ by the all genus $\mathcal{F} = \log|\mathcal{Z}_{\rm top}|^2$ in (3.14) and (3.15)) can be neglected in our $\Lambda \to \infty$ scaling limit?

The puzzle is resolved by considering the product representation (1.17) of $\mathcal{Z}_{\rm top}$, and noting that all contributions of curves with nonvanishing charge q are exponentially suppressed as $e^{-\Lambda}$ since $J \sim \Lambda$. This leaves only the MacMahon function (1.21), which since $g_{\rm top}$ does not scale with Λ gives only a finite contribution to the entropy, independent of Λ. Hence for $\Lambda \to \infty$, this contribution can indeed be neglected.

Incidentally, the same kind of reasoning resolves the puzzle why the D4D2D0 entropy in the large D0-charge limit does not receive enormous R^2 corrections, despite the fact that $g_{\rm top} \to \infty$ when $|q_0| \to \infty$. Again, this this system becomes weakly curved in the M-theory description, and again all corrections are manifestly suppressed when using the product formula for $\mathcal{Z}_{\rm top}$.

Now, having convinced ourselves that our solutions and the entropy computed from them are reliable, we face a puzzle. The OSV conjecture is supposed to be valid precisely at large Λ. But its prediction for the leading asymptotic of $\ln\Omega$ is, by construction, the single centered black hole entropy, which scales as Λ^2, not Λ^3. So how can this be compatible with what we find here?

Despite the obvious tension this creates, this does not immediately mean the OSV conjecture is wrong. There are two important subtleties. The first one is that $\Omega(\Gamma)$ is an *index*, the second is that the Λ^3 scaling holds at $J = i\infty$ but for example not at the attractor point of Γ, where two centered solutions do not exist.

To address these subtleties, we need a better understanding of various types of composite BPS states, as well as the computation of their contributions to the index, which we do in the following sections. We postpone further discussion to section 7.

3.6 D6-D0 bound states

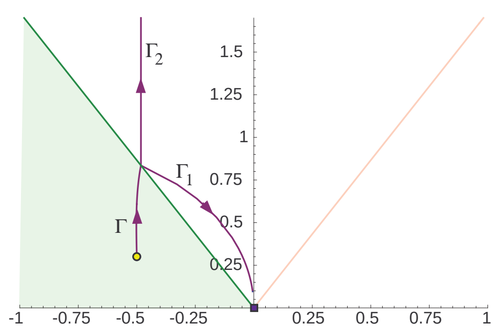
At large volume and zero B-field, D6 and D0 branes do not form BPS bound states. However this can change when the B-field is sufficiently large [83, 84, 85], or equivalently when a sufficiently large $U(1)$ flux is turned on on the D6. Let

$$\Gamma_1 = (p^0, 0, 0, 0), \quad \Gamma_2 = (0, 0, 0, q_0), \quad \Gamma = \Gamma_1 + \Gamma_2. \tag{3.53}$$

Then $Z_{\rm hol}(\Gamma) = p^0 (B + iJ)^3/6 - q_0$, which clearly has a zero locus in the interior of moduli space, so no single centered BPS black hole solutions exist. To be more explicit, let us take for example as in the previous subsection

$$B + iJ = zP/|p^0| \tag{3.54}$$

with $z = x+iy \in \mathbb{C}$ and P some positive class in $H^2(X, \mathbb{Z})$, and write $q_0 = \rho\, P^3/(p^0)^2$. Then up to an overall constant positive factor $P^3/(p^0)^2$ we have $Z_1 = \mathrm{sign}(p^0)\, z^3/6$, $Z_2 = -\rho$, and $Z = \mathrm{sign}(p^0)\, z^3/6 - \rho$, which has a zero in the upper half z-plane.



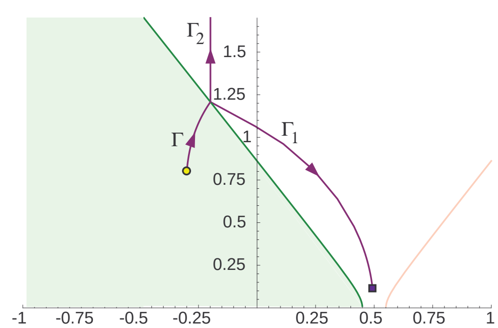


Figure 4: Left: attractor flow tree in z-plane for $p^0 > 0$, $\rho = -1$, $z_\infty = -0.5 + 0.3\, i$. The shaded region on the left is the stable region, in which the BPS state exists. It is bounded by the marginal stability line (green line). The light pink line on the right is the line of anti-marginal stability (where the phases anti-align). The D0-attractor flow (Γ_2) continues up to $\mathrm{Im}\, z = \infty$. **Right**: Analogous plot for Γ_1 defined as for fig. 2a and $\Gamma_2 = (0, 0, 0, -1)P^3/r^2$ with P and r as for fig. 2.

To see if there is a bound state of Γ_1 and Γ_2 in some region of moduli space, it is sufficient to check if a marginal stability wall exists, since we know that the constituents themselves, the D0 and the D6, exist everywhere in moduli space (at least in the large volume region we are considering). It is easy to see that $\mathrm{Im}(Z_1 \bar{Z}_2) = 0$ when $y = \sqrt{3}|x|$. To have the phases align rather than anti-align on this line, we moreover need $\mathrm{Re}(Z_1 \bar{Z}_2) > 0$, i.e. $\mathrm{sign}(p^0)\, \rho\, x > 0$. To see which side of this line is stable, we can use (3.23), which gives $|x| > |y|/\sqrt{3}$. Taking into account that only a true line of marginal stability can bound a stability domain, we get as our final result for the zone in the upper half z-plane where there exists a stable D6-D0 BPS bound state:

$$|\mathrm{Re}\, z| > \mathrm{Im}\, z/\sqrt{3}, \qquad \mathrm{sign}(\mathrm{Re}\, z) = \mathrm{sign}(p^0 q_0). \tag{3.55}$$

This is illustrated in fig. 4a. Similar (but mathematically slightly more complicated) considerations hold when the D6 is replaced by a more general D6-D4-D2-D0 brane; an example is shown in fig. 4b. Note that when the D6-D4-D2-D0 has a nonzero entropy, it can also "absorb" some of the D0 inside its horizon. The amount of D0-brane charge which can be absorbed in this way is always bounded however, as can be seen for example from (3.19), which always goes negative when $n \to \infty$. This is another example of multiple BPS realizations of the same charge. Again, recall that according the the split attractor flow conjecture the number of possibilities is bounded.

3.7 Sun-Earth-Moon systems

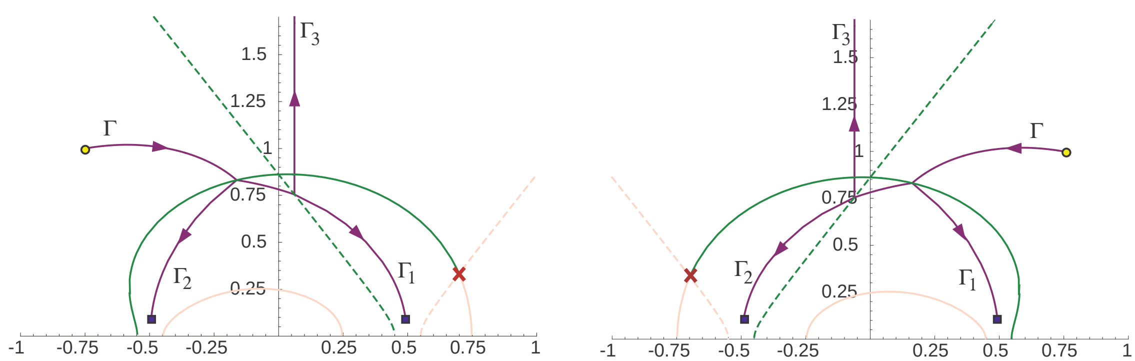


Figure 5: Three-legged flow trees with Γ_1 and Γ_2 as for fig. 2a and $\Gamma_3 = (0, 0, 0, -0.01)P^3/r^2$. On the left $z_\infty = -0.75 + i$, on the right $z_\infty = 0.75 + i$. Note that this choice affects the order of the splittings: on the left we have $\Gamma \to (\Gamma_2, \Gamma_1 + \Gamma_3)$ followed by $\Gamma_1 + \Gamma_3 \to (\Gamma_1, \Gamma_3)$, while on the right Γ_1 splits off first. The solid green line indicates the marginal stability line for the first split, the dotted green line for the second. The light pink lines are the corresponding anti-marginal stability lines. The red cross indicates a zero of the central charge of the intermediate charge ($\Gamma_1 + \Gamma_3$ on the left), implying that this charge does not have a single centered realization.

We can also combine this sort of bound state with another state to produce a state with overall D6 charge zero. For example we can dress the (Γ_1, Γ_2) solutions of section 3.4 with a $\overline{D0}$-brane bound to one of the centers. The corresponding attractor flow trees are illustrated in fig. 5. Note that the charge to which the $\overline{D0}$ binds depends on the choice of B-field at infinity, i.e. $\mathrm{Re}\, z_\infty$. The transition between the two occurs when the initial attractor flow (for charge Γ) hits the point where all three phases of the $Z(\Gamma_i)$ align, that is at the intersection point of the dotted and solid green lines in the figure. In the case at hand, this happens when z_∞ crosses the imaginary axis. When z_∞ is exactly on the imaginary axis, Γ_3 (a $\overline{D0}$) is at most marginally bound: its phase lines up there with the phase of $\Gamma_1+\Gamma_2$ (a $D4+D0$), and there is no energetic obstruction to taking away Γ_3 from $\Gamma_1+\Gamma_2$ as far as one wishes.

For positive D4-charge P, it is not possible to construct such bound states with Γ_3 a $D0$-brane rather than $\overline{D0}$-brane; in the figure above, this would essentially flip the marginal and anti-marginal stability lines involving Γ_3, so after the first split one would be outside of the stable region for the remaining bound state involving Γ_3. This corresponds to the fact that only $\overline{D0}$-branes, not $D0$-branes, form bound states with D4-branes in our sign conventions.

Figure 6: Left: Sketch of a BPS Sun - Earth - Moon configuration in space. **Right:** Quiver diagram representing intersection products $I_{ij}=\langle\Gamma_i,\Gamma_j\rangle$ between the centers. In the case at hand $I_{21}, I_{13}, I_{32} > 0$. For the $D6$-$\overline{D6}$-$\overline{D0}$ system, the microscopic quiver would look identical except for an additional multiplicity 3 arrow going from the $\overline{D0}$ node to itself, representing the three moduli corresponding to the $\overline{D0}$ moving around in X.

The supergravity solutions representing these bound states are Sun-Earth-Moon configurations, as shown in fig. 6a. The positions of the centers $\vec{x}_i$, $i=1,2,3$ are constrained by the integrability conditions (3.21):

$$\frac{I_{13}}{R_{13}} - \frac{I_{21}}{R_{21}} = \theta_1 \qquad + \text{ cycl. perm.} \tag{3.56}$$

where $I_{ij}=\langle\Gamma_i,\Gamma_j\rangle$, $R_{ij}=|\vec{x}_i-\vec{x}_j|$, $\theta_i = 2\mathrm{Im}(e^{-i\alpha}Z_i)_\infty$. Note that $\theta_1+\theta_2+\theta_3=0$ and the third equation is just the sum of the first two. More concretely in the case at hand we can take say Γ_1 to be a pure D6 with $U(1)$ flux $F=S_1$ turned on, Γ_2 the anti-brane of a pure D6 with flux $F=S_2$ turned on, and Γ_3 a charge $-n$ anti-D0 brane, so according to (A.3) we have

$$\Gamma_1 = e^{S_1}(1+\frac{c_2}{24}), \quad \Gamma_2 = -e^{S_2}(1+\frac{c_2}{24}), \quad \Gamma_3 = -n\,\omega. \tag{3.57}$$

In this case, $I_{13}=n$, $I_{32}=n$, and $I_{21}=\frac{P^3}{6}+\frac{c_2\cdot P}{12}=I_P$, where $P=S_1-S_2$ is the total D4-brane charge. [28]

[28] If we wanted to establish the existence of these multicentered solutions directly in supergravity without

Note that depending on the sign of $\theta_3 \sim -\sin\alpha$, which is determined by the value of the B-field, R_{23} is smaller or larger than R_{31}, corresponding to the anti-D0 binding to the D6 or to the anti-D6. When $\theta_3 = 0$, the anti-D0 moves on a plane equidistant from the D6 and the anti-D6 center, so it can escape to infinity. Indeed, at this locus in moduli space (which includes zero B-field in the case of zero total D2-charge), the bound state between an anti-D0 and a D4 ($=\Gamma_1 + \Gamma_2$) is only marginal.

3.8 Scaling solutions

Considering the R_{ij} as independent variables, the equations (3.56) always have a scaling solution

$$R_{ij} \to \lambda I_{ij}, \qquad \lambda \to 0, \tag{3.58}$$

independent of the θ_i. In the limit $\lambda = 0$, the coordinates of the 3 centers coincide and hence the solution becomes indistinguishable from a single centered black hole solution to a distant observer. (However, for an observer remaining close to the centers, they actually stay at finite distance: Within a coordinate distance of order λ from the centers $H(\vec{x}) \sim \lambda^{-2}$ and hence $e^{-2U(\vec{x})} \sim \lambda^{-2}$, so the presence of the warp factor in (3.7) implies that the observer remains at an order one geodesic distance. What is happening is that a throat is developing and the observer disappears down the throat.)

However, the R_{ij} are actually not quite independent: they equal the lengths of the edges of a triangle in flat space, and as such must satisfy the triangle inequality. Thus a necessary and sufficient condition for the scaling solution to (3.56) to exist is

$$I_{21} + I_{13} \geq I_{32} \qquad + \text{ cycl. perm.} \tag{3.59}$$

In the case at hand this reduces to $n \geq \frac{1}{2} I_P = \frac{1}{12} P^3 + \frac{1}{24} c_2 P$. In particular, this implies that the total charge is necessarily nonpolar, since $\hat{q}_0 = \frac{1}{24} P^3 + \frac{1}{24} c_2 P - n < 0$. This is compatible with general expectations, as only nonpolar states should be able to form black holes.

Since the scaling solution is independent of the θ_i, the branch of the solution moduli space to (3.56) continuously connected to the black hole in this way will never decay when the θ_i (in other words the background Kähler moduli) are varied. It is therefore represented by a single centered attractor flow rather than an attractor flow tree.

Conversely, if the triangle inequalities (3.59) are *not* satisfied, then the solutions can always be forced to decay by varying the θ_i (so the solution is described by an attractor flow tree).

To see this, let us assume that one of the triangle inequalities is violated. Without loss of generality we can take $I_{21} > I_{13} + I_{32}$. Then we claim that when $\theta_3 > 0$, taking θ_1 to zero will necessarily force $\vec{x}_1$ to separate infinitely far from $\vec{x}_2$ and $\vec{x}_3$, and when $\theta_3 < 0$,

invoking the split attractor flow conjecture, we would have to check that $H(\vec{x})$ lies in $\operatorname{dom} S$ for all $\vec{x}$. This is difficult. One can show (dropping c_2 corrections) that when the integrability conditions are satisfied, the discriminant of $H(\vec{x})$ goes to a positive constant at infinity and goes to $+\infty$ near each of the three centers, and therefore takes on its minimal value at some finite point in $\mathbb{R}^3$. If this point is on an axis of symmetry then one can further show rigorously that $\mathcal{D}(H(\vec{x}))$ is bounded below by a positive constant.

taking θ_2 to zero will similarly separate $\vec{x}_2$ from $\vec{x}_1$ and $\vec{x}_3$. Let us consider the $\theta_3 > 0$ case, the other case is analogous. When $\theta_1 = 0$, we then have $\theta_2 = -\theta_1 - \theta_3 = -\theta_3 < 0$ and the equilibrium conditions (3.56) imply *either*

$$R_{13} = R_{21} = \infty, \quad R_{23} = -\frac{I_{32}}{\theta_2} \tag{3.60}$$

which corresponds to the claimed infinite separation, *or*

$$R_{13} = \lambda I_{13}, \quad R_{21} = \lambda I_{21}, \quad R_{32} = \lambda' I_{32}, \quad \text{where } \lambda' < \lambda < \infty. \tag{3.61}$$

However, since we must have $R_{21} \leq R_{13} + R_{32}$ to have an actual solution, the above gives $I_{21} < I_{13} + I_{32}$, contradicting the initial assumption. Hence (3.60) remains the only possibility; the solution is forced to split at $\theta_1 = 0$ (with the stable side being $\theta_1 < 0$, as a slight extension of the analysis shows).

All this is of course in perfect agreement with what we expect from the attractor flow picture, as well as with general expectations for polar states.

In section 5, we will study similar bound states both in the spacetime picture and in the microscopic quiver picture. We will see that the BPS index factorizes precisely when the inequalities (3.59) are violated. Moreover, the BPS index undergoes some sort of phase transition — no longer factorizing and starting to grow exponentially — as soon as (3.59) are satisfied. Note this is exactly where the black hole branch opens up. Hence, this qualitative change is physically expected from the spacetime picture, but highly nontrivial from the microscopic point of view.

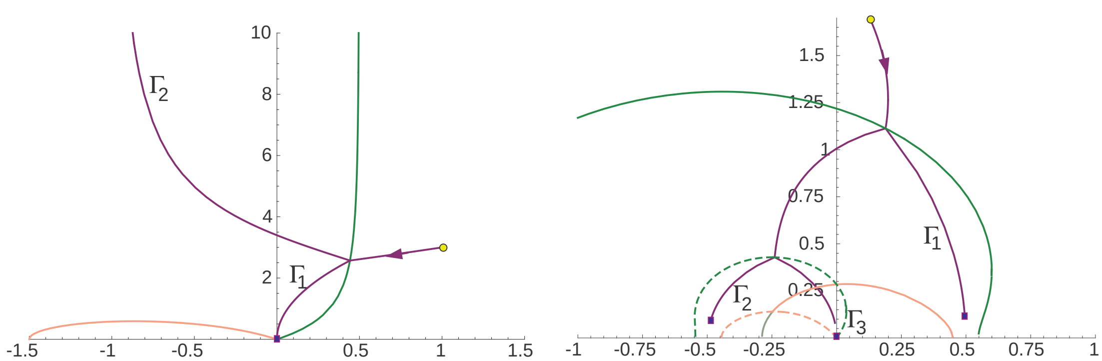


Figure 7: Two more bound states with total D6-brane charge equal to 1. **Left:** $\Gamma_1 = D6$, $\Gamma_2 = -D2 + D0$. Note that the line of marginal stability goes up along a vertical asymptote all the way to infinite radius. **Right:** Γ_1 and Γ_2 chosen as in fig. 2 (carrying 1 resp. -1 unit of D6 charge), and $\Gamma_3 = D6$.

3.9 Even more complicated multicentered bound states

One can imagine many other multicentered configurations involving various charges, for example we can add more anti-D0 "moons", or replace the D0 particles by D2-D0 particles.

Some examples with net D6 charge 1 are shown in fig. 7. These can in turn be used as building blocks for the D4-D2-D0 bound states of interest, and so on, even leading to fractal-like flow-trees, as shown in fig. 8.

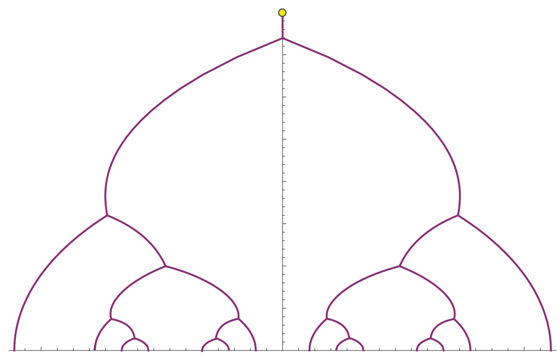


Figure 8: The D6 split flows of fig. 7*b* (and their conjugates) can be iteratively combined to form fractal-like flow trees with zero total D6 charge. An example is shown with 14 pure (fluxed) D6 / anti-D6 centers. It is possible to write compact analytic formulae describing these fractal flow families.

Enumerating this zoo and taking into account all existence conditions, let alone computing their BPS ground state degeneracies, would appear very hard, to say the least. However, to count degeneracies of polar states, we can use the fact that these are guaranteed to split in two clusters at a wall of marginal stability somewhere in moduli space. The problem is then reduced to computing the degeneracies of the two individual clusters. This might still be complicated if one wants to compute exact expression for the the degeneracies based on enumerating all possible further splits corresponding to the structure of the clusters (although one could imagine working recursively), but in suitable cases, it is possible to circumvent this problem. The idea is to go to a regime in which the most significant contributions to the fareytail series come from the polar terms corresponding to flow trees which initially split in branes with charges Γ_1 and Γ_2 with D6-charge 1 resp. -1. The indices of BPS states for such branes turn out to be more or less given by rank one DT invariants. (The precise relation is explained in section 6.1.2 and section 6.3.2.) This will allow us to express the BPS indices of the relevant polar terms in the fareytail series in terms of the DT invariants, along the lines of section 3.1. There is no need to consider further splits of the flow tree, since the DT invariants already count all BPS states of the two initial brane constituents. Using the relation between DT and GW invariants reviewed in section 1.3, we will thus be led to an expression of $\mathcal{Z}_{\rm BH}$ in terms of the topological string partition function, and to the OSV conjecture.

Finally, we note here that although a priori we should also consider splits in two charges both of which have $p^0 = 0$, those are easily shown to be absent for flows coming from large J. Consider a charge $\Gamma = (0, P, 0, q_0)$ and a candidate split in $\Gamma_1 = (0, P_1, Q_1, q_{0,1})$, $\Gamma_2 = (0, P_2, -Q_1, q_{0,1})$, and let us take $B_\infty = 0$, $J_\infty = y\, P$, $y \to \infty$. Then

$$\langle \Gamma_1, \Gamma_2 \rangle \, \mathrm{Im}(Z_1 \bar{Z}_2) = -(P \cdot Q_1) \left(\frac{1}{2} P^3 (P \cdot Q_1) y^3 + O(y) \right) \leq 0, \tag{3.62}$$

so (3.23) is not satisfied. (When J is not proportional to P it is perfectly possible to have a $D4D2D0$ split into a pair of $D4D2D0$ states at $B_\infty = 0$ and large J_∞.)

4. Microscopic description

States corresponding to flow trees have a microscopic description as well. This will be the subject of the present section.

4.1 D-brane picture at $g_s = 0$

The microscopic D-brane picture is valid at $g_s|\Gamma| \ll 1$ with $|\Gamma|$ some appropriate measure of the "size" of the charge Γ. It describes the state as an object sitting at a single point in the noncompact space. The macroscopic picture, valid in the opposite regime $g_s|\Gamma| \gg 1$, at first sight looks very different, with bound states looking like atoms or molecules rather than D-branes geometrically glued together. Nevertheless, the two pictures can be shown to transform smoothly into each other when varying g_s; for a detailed analysis see [65].

In the large radius limit, IIA D-branes are well described by holomorphic geometrical objects wrapped around various even dimensional cycles. The F-term constraints determining the moduli spaces of these objects do not receive α' corrections [42]. On the other hand, the D-term constraints, which govern stability and decay, *do* receive important α' corrections [42]. As a result, phenomena such as decay at marginal stability at some finite value of the Kähler moduli tend to be invisible in the IIA large radius geometrical description. However, there is a simple universal microscopic picture which does capture this phenomenon accurately. This is originally due to [83] and has been extended in many works on the categorical description of D-branes (as reviewed in [66]).

This goes roughly as follows. Let us consider a bound state of two D-branes with charges Γ_1 and Γ_2. When near the wall of marginal stability (i.e. when the phases α_i of the central charges $Z(\Gamma_i)$ are almost identical), there are light bosonic open string states stretching between the D-branes corresponding to Γ_1 and Γ_2, whose mass squared equals [86]

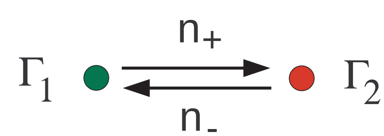


Figure 9: Bound state quiver

$$m^2 \sim q\,(\alpha_2 - \alpha_1), \qquad q = \pm 1. \tag{4.1}$$

More precisely this is the tree level mass squared in the mirror intersecting D3-brane picture when the D3-branes are at the same point in $\mathbb{R}^3$, the light strings corresponding to string localized at the intersection points.[29] In the low energy description of the D-branes as a supersymmetric quantum mechanical system with 4 supercharges, the light strings appear as chiral multiplets Φ_i (dimensionally reduced to $d = 1$), represented by the arrows of a quiver[30] with two nodes as in fig. 9, and (4.1) can be understood as being induced by a D-term potential [83]

$$V(\phi) = \frac{1}{2\mu} D^2, \qquad D = \sum_a q_a |\phi_a|^2 - \mu(\alpha_2 - \alpha_1). \tag{4.2}$$

where the $q_a = \pm 1$ are charges with respect to the relative $U(1)$ between the branes and μ is some constant (specified below in section 4.2). The fermionic superpartners of the

[29] If the branes are not at the same point in $\mathbb{R}^3$ there is an additional mass term $m^2 \sim |\vec{x}_1 - \vec{x}_2|^2$ and supersymmetry is generically broken at $g_s = 0$.

[30] This quiver should be interpreted in a loose sense in the present discussion. In particular we allow the branes correspondng to the nodes to have nontrivial moduli spaces here, not necessarily realized by simple adjoint fields as in the proper definition of a quiver. These moduli spaces might for example arise from lumping together several standard quiver nodes into one.

ϕ_i remain massless at tree level, but when both positive and negative q_i are present, disc instantons ending on the D3-branes can produce a nontrivial superpotential depending on the ϕ^i, lifting pairs of massless fermions of opposite charges, but leaving the difference of $q = \pm 1$ massless fermions invariant. If we denote the number of stretched strings with $q = \pm 1$ by $n_\pm$, then we have for the the index

$$n_+ - n_- = \langle \Gamma_1, \Gamma_2 \rangle \tag{4.3}$$

with $\langle \Gamma_1, \Gamma_2 \rangle$ the symplectic intersection product between the charges. In type IIB this is just the geometric intersection product between the D3-branes, n_+ (n_-) being the number of positive (negative) intersection points. For IIA we define this product in appendix A.

From (4.1) we see that when we are close to the marginal stability wall, on the side where

$$\langle \Gamma_1, \Gamma_2 \rangle (\alpha_1 - \alpha_2) > 0, \tag{4.4}$$

there will always be tachyonic strings present stretching between the constituent branes. Condensation of these tachyons produces a BPS bound state of total charge Γ. Since we assumed the state decays when crossing the wall, no such tachyons exist on the other side of the wall, where the above expression becomes negative. This will indeed be the case when either $n_+ = 0$ or $n_- = 0$ (possibly effectively after lifting pairs by F-term masses). Note that this stability condition is identical to the supergravity condition (3.23) for small $\alpha_1 - \alpha_2$.

When we have two single D-branes, hence a gauge group $U(1) \times U(1)$, with respective deformation moduli spaces $\mathcal{M}_1$ and $\mathcal{M}_2$, and if say $n_- = 0$ over all of $\mathcal{M}_1 \times \mathcal{M}_2$, then the moduli space $\mathcal{M}$ of the bound state will be a $\mathbb{CP}^{|\langle \Gamma_1, \Gamma_2 \rangle| - 1}$ fibration over $\mathcal{M}_1 \times \mathcal{M}_2$, with the $\mathbb{CP}^{|\langle \Gamma_1, \Gamma_2 \rangle| - 1}$ fiber coming from solving the D-flatness condition $D = 0$ and modding out by $U(1)$. If the fiber does not degenerate anywhere, the Euler characteristic factorizes as

$$\chi(\mathcal{M}) = \chi(\mathbb{CP}^{|\langle \Gamma_1, \Gamma_2 \rangle| - 1})\, \chi(\mathcal{M}_1)\, \chi(\mathcal{M}_2) = |\langle \Gamma_1, \Gamma_2 \rangle|\, \chi(\mathcal{M}_1)\, \chi(\mathcal{M}_2). \tag{4.5}$$

Identifying the Euler characteristic (up to a sign) with the index of supersymmetric states Ω, this gives a corresponding factorization of Ω.

As described in section 3.2.3, attractor flow trees provide a useful canonical prescription for an iterated assembly or decay process of multicentered configurations, in particular because stability is guaranteed to be preserved when moving upstream along an attractor flow, and decay, whenever possible, is guaranteed to occur at some point when flowing down. Instead of splitting or joining (clusters of) centers in supergravity, we may equally well think microscopically at $g_s = 0$ and split or glue branes together through tachyon condensation as described above, while following the same flow trees. This makes sense since flow trees are determined entirely by central charges, which are universal, exact data, independent of the picture in which one is working. In this way attractor flow trees continue to be meaningful even microscopically.

Microscopic counterparts of (3.23) exist in the framework of the derived category as well. For a nice discussion of how it appears for bound states of holomorphic vector bundles and its relevance to the question of existence of stable vector bundles, see [100].

4.2 Quiver description of bound states

Building on the reasoning outlined above, one finds that quivers give a low energy, weak string coupling description of bound states of simple, rigid objects (such as D6 or anti-D6 branes carrying $U(1)$ flux), near a locus in moduli space where the central charges of the objects all line up. Let us quickly review some useful facts about this representation, referring to [65] for more details.

In a region where the phases almost line up, the objects are almost mutually supersymmetric, and there will be open strings stretched between them whose lightest fermionic modes are massless and whose lightest bosonic modes have squared masses proportional to the phase differences of the central charges, along the lines sketched above (this is assuming the objects coincide in the noncompact space). The system can be modeled at low energies by quiver quantum mechanics, obtained by dimensionally reducing the corresponding $\mathcal{N}=1$, $d=4$ quiver gauge theory. The multiplicities of the objects associated to the nodes i are given by the dimension vector d_i. The degrees of freedom of the quantum mechanics are the (possibly nonabelian) positions of the nodes in the noncompact space and the $U(d_i) \times U(d_j)$ complex bifundamental scalars ϕ_{ij}^a, $a = 1, \ldots, K_{ij}$, associated to the light open strings from node i to node j, plus their fermionic partners. When $g_s \to 0$ keeping other parameters fixed, the supersymmetric ground state wave functions live on the Higgs branch, with all node positions coincident and the bifundamental vevs subject to the D-term constraints

$$\sum_j \sum_a (\phi_{ij}^a)^\dagger \phi_{ij}^a - \sum_j \sum_a \phi_{ji}^a (\phi_{ji}^a)^\dagger = \vartheta_i \, \mathbf{1}_{d_i} \quad \forall i. \tag{4.6}$$

The Fayet-Iliopoulos parameters ϑ_i are given by the background moduli as

$$\vartheta_i = 2 m_i (\alpha_i - \alpha_0) \tag{4.7}$$

where $\alpha_i = \arg Z_i$, $m_i = |Z_i|$ and $\alpha_0 = \sum_i d_i m_i \alpha_i / \sum_i d_i m_i$, with Z_i the normalized central charge of the ith node. Note that $\sum_i d_i \vartheta_i = 0$, and that the condition for all the phases to almost line up is $\vartheta_i \ll 1$. If closed oriented loops are present, there can be a superpotential $W(\phi)$ as well. If there are no such closed loops, gauge invariance prohibits a nonzero W. The quiver moduli space is thus given by

$$\mathcal{M} = \{\phi \,|\, (4.6) \text{ satisfied, and } \partial W = 0\} \,/\, U(d_1) \times \cdots \times U(d_n). \tag{4.8}$$

So far we kept the branes at the same point in the noncompact space. However when one takes the objects apart (including splitting the nodes with multiplicity $d_i > 1$ in d_i separate branes, labeled by an index $\alpha = 1, \ldots, d_i$), the stretched strings become massive and can be integrated out. At one loop this produces a potential on position moduli space, with supersymmetric minima at

$$\sum_{j,\beta} \frac{I_{ij}}{|\vec{x}_{i\alpha} - \vec{x}_{j\beta}|} = \vartheta_i \quad \forall i, \tag{4.9}$$

where $I_{ij} = K_{ij} - K_{ji} = \langle \Gamma_i, \Gamma_j \rangle$. When the solutions to this equation have separations $|\vec{x}_i - \vec{x}_j|$ which are sufficiently large, the procedure of integrating out the stretched strings is self-consistent. Depending on the parameter regime, the supersymmetric ground state wave functions will peak on the "Coulomb branch" ($\phi = 0$, $\Delta\vec{x} \neq 0$) or on the "Higgs branch" ($\phi \neq 0$, $\Delta\vec{x} = 0$), thus interpolating between the two pictures of bound states [65]. Note that equation (4.9) is almost exactly the same as the supergravity position constraint equations (3.21):

$$\sum_{j,\beta} \frac{I_{ij}}{|\vec{x}_{i\alpha} - \vec{x}_{j\beta}|} = \theta_i \quad \forall i, \tag{4.10}$$

where $\theta_i = 2\,\mathrm{Im}(e^{-i\alpha} Z_i) = 2m_i \sin(\alpha_i - \alpha)$, $\alpha = \arg(\sum_i m_i e^{i\alpha_i})$, $\sum_i d_i \theta_i = 0$. The identity of the form of these equations despite being in very different regimes is due to a non-renormalization theorem. Note furthermore that in the strict physical domain of validity of the quiver picture, we have $\vartheta_i \ll 1$, so $\alpha_0 \approx \alpha$ and $\vartheta_i \approx \theta_i$. When moving away from the locus where all phases line up, ϑ_i and θ_i start to deviate; this should not come as a surprise, since the value of constants on the right hand side of the one loop result (4.9) are not protected and will receive corrections.

Thus we see that in the quiver description of bound states, the correspondence between multicentered solutions and microscopic bound states is rather explicit.

4.3 Geometrical relations between D4 and D6-anti-D6 bound states

IIA D-brane bound states are rather well understood in the $J \to \infty$, $g_{\mathrm{IIA}} \to 0$ limit, where they are essentially given by holomorphic vector bundles, or more generally coherent sheaves. In this geometric description, F-term constraints do not receive α' corrections, but D-term constraints do. D-terms govern stability, and as a result many decay phenomena are completely invisible at large radius from the microscopic point of view. This is not universally true, since μ-stability can be seen at large radius. However, decays of the kind we have investigated such as a D4 splitting into a D6 and anti-D6 are not detectable if one limits one's attention to holomorphic vector bundles on holomorphic 4-cycles.

In spite of all this, in this section we will nevertheless arrive at a picture for (sufficiently polar) D4-D2-D0 brane states in the language of holomorphic sheaves which tantalizingly hints at the "split" nature of the corresponding BPS states. In particular, although in the geometrical regime we cannot literally see those states split in the D6 and anti-D6 branes which are their building blocks according to the split flow picture, a lot of the structure of their moduli spaces is suggestive of this structure.

The picture we develop here was first proposed in [44] and exploited further in [47]. We review it here for completeness and add a number of observations. The picture we arrive at is heuristic and will not be used in the proof of the OSV formula. It is nevertheless a source of very useful intuition.

If the divisor Σ in the class P is *frozen* at $\Sigma = \Sigma_0$, the moduli space of BPS configurations reduces to $\mathrm{Hilb}^N \Sigma_0$ [87, 88, 89, 3, 90], i.e. the Hilbert scheme of N points on Σ_0, and by (2.7), since $\dim \mathrm{Hilb}^N \Sigma_0 = N \dim \Sigma_0$ is always even,

$$d_{\Sigma_0}(F, N) = \chi(\mathrm{Hilb}^N \Sigma_0). \tag{4.11}$$

The generating function for these Euler characteristics is given by Göttsche's formula [92] (see [90] for a pedagogical review)

$$\sum_N \chi(\mathrm{Hilb}^N \Sigma_0)\, q^N = \prod_{n \geq 1} (1-q^n)^{-\chi(\Sigma_0)}. \tag{4.12}$$

However, in reality, the divisor Σ is not some fixed Σ_0, but has a deformation moduli space, and even when a sufficiently generic flux is turned on such that all deformation degrees of freedom are frozen by the condition $F^{2,0} = 0$, there might be several such isolated points in the divisor moduli space. Moreover, we need to sum over different fluxes giving the same total charge. In the limit $N \to \infty$, all those extra degrees of freedom only give subleading contributions to the entropy, but at smaller N, in particular for the polar states, this is not the case.

One could try to correct this by considering the full moduli space, say for F a flux pulled back from $H^2(X)$ (such that none of the deformation moduli of P are obstructed), as a fibration over $\mathcal{M}_P = \mathbb{CP}^{I_P - 1}$ with fiber given by $\mathrm{Hilb}^N P$. If the fibration has no singular fibers, the orbifold Euler characteristic of the total space would just be the product of $\chi(\mathcal{M}_P) = I_P$ and $\chi(\mathrm{Hilb}^N P)$, and the generating function would be obtained simply by multiplying (4.12) by I_P. A simple example shows this idea to be too naive: Consider the moduli space with one pointlike instanton. This fibers over X with fiber $\mathbb{C}P^{I_P - 2}$, and hence the Euler character is $\chi(X)(I_P - 1)$. The reason for the discrepancy is the presence of a complicated, self-intersecting locus in $\mathcal{M}_P$ where the fiber P becomes singular, so the simple factorization formula does not hold. Figuring out the correct formula in this picture appears very hard.

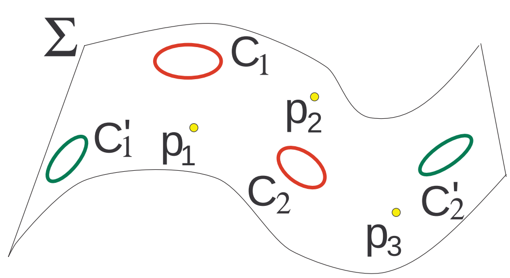


Figure 10: Irreducible curves C_k, $C'_{k'}$ and points p_i contained in divisor Σ.

An alternative way of thinking about the moduli space, at least for sufficiently large P and sufficiently small N and F (i.e. sufficiently polar states) is as follows (see also [44, 47]). Write as in (2.37) $F = \frac{P}{2} + f^{\parallel} + \gamma + f^{\perp}$. Recall that supersymmetry requires $F^{0,2} = (\gamma + f^{\perp})^{0,2} = 0$, which is equivalent to the statement that $\gamma + f^{\perp}$ is Poincaré dual to a collection of holomorphic 2-cycles on Σ. Note that this puts restrictions on the divisor deformation moduli, since generically the divisor will not contain curves other than those obtained by intersecting other divisors (which correspond to $f^{\parallel}$). More precisely we have

$$F = \iota_\Sigma^* S + [C]_\Sigma - [C']_\Sigma \tag{4.13}$$

where $S \in \frac{P}{2} + H^2(X,\mathbb{Z})$, $\iota_\Sigma^* S = \frac{P}{2} + f^{\|}$, C and C' are collections of holomorphic curves, and $[\cdot]_\Sigma$ denotes the corresponding (co)homology class on Σ. (Note that this formula suffers from the ambiguity $C \to C + C''$, $C' \to C' + C''$, which is one of the reasons why the picture developed here is rather heuristic.) Hence we can build supersymmetric configurations by first picking a set of points p_i, $i = 1, \ldots, N$, and a collections of holomorphic curves C, C' in X, and require our divisor Σ to contain all of those (see fig. 10). This is possible when the number of points and curves (and their degrees) is sufficiently small compared to the number of deformation moduli of Σ. For example for the hyperplane $\sum_n a_n x_n = 0$ in the Fermat Quintic $Q := \sum_n x_n^5 = 0$, requiring the curve $x_1 = -x_2$, $x_3 = -x_4$, $x_5 = 0$ to lie in the hyperplane puts $a_1 = a_2$, $a_3 = a_4$, reducing the moduli space from $\mathbb{CP}^4$ to $\mathbb{CP}^2$.

The adjunction formula for irreducible holomorphic curves C on Σ gives $2\,\chi_h(C) = -C^2 - K_\Sigma \cdot C$ where χ_h is the holomorphic Euler characteristic, i.e. one minus the genus of the curve. We can also write $K_\Sigma \cdot C = \int_C P := P \cdot [C]$ where the first intersection product is on Σ and the last on X. By additivity of the Euler characteristic, this formula extends to collections of holomorphic curves. Using this, the charges (2.1) and (2.4) can be computed as

$$
\begin{aligned}
q_A &= D_A \cdot ([C] - [C'] + P \cdot S) && (4.14)\\
q_0 &= \frac{P^3 + c_2 \cdot P}{24} - N + \frac{1}{2} P S^2 + S \cdot ([C] - [C']) && (4.15)\\
&\quad -\chi_h(C) - \frac{P}{2} \cdot [C] - \chi_h(C') - \frac{P}{2} \cdot [C'] - [C] \cdot [C'] && (4.16)
\end{aligned}
$$

All intersection products are on X, except for the last term, which is an intersection of two curves within Σ.

Now let us compare this to the charges of a bound state of the kind described in the previous sections.

Start with a single D6 brane containing a BPS "gas" of D2- and D0-branes, with D2-charge $-\beta_1 \in H_2(X,\mathbb{Z})$ where β_1 is an effective curve class,[31] and D0-charge $n_1 \in \mathbb{Z}$. Now add D4-brane charge by turning on a flux S_1, giving according to (A.3) a total charge

$$
\begin{aligned}
\Gamma_1 &= e^{S_1}(1 - \beta_1 + n_1\,\omega)(1 + \frac{c_2(X)}{24}) && (4.17)\\
&= \left(1,\, S_1,\, \frac{S_1^2}{2} - \beta_1 + \frac{c_2}{24},\, \frac{S_1^3}{6} - \beta_1 S_1 + \frac{c_2}{24} S_1 + n_1\right). && (4.18)
\end{aligned}
$$

Do the same for a second D6-brane and take its charge conjugate, so

$$
\begin{aligned}
\Gamma_2 &= -e^{S_2}(1 - \beta_2 + n_2\,\omega)(1 + \frac{c_2(X)}{24}) && (4.19)\\
&= \left(-1,\, -S_2,\, -\frac{S_2^2}{2} + \beta_2 - \frac{c_2}{24},\, -\frac{S_2^3}{6} + \beta_2 S_2 - \frac{c_2}{24} S_2 - n_2\right). && (4.20)
\end{aligned}
$$

Defining

$$
\tilde{P} := S_1 - S_2, \qquad \tilde{S} := \frac{S_1 + S_2}{2}, \tag{4.21}
$$

[31] In our conventions, D6-branes form BPS states with *anti*-D2 branes.

the total charge $\Gamma = \Gamma_1 + \Gamma_2$ can be written as

$$\Gamma = \left(0,\, \tilde{P},\, \beta_2 - \beta_1 + \tilde{P}\tilde{S},\, \frac{\tilde{P}^3 + c_2\tilde{P}}{24} + \frac{1}{2}\tilde{P}\tilde{S}^2 + \tilde{S}(\beta_2 - \beta_1) - n_2 - \frac{\tilde{P}}{2}\beta_2 + n_1 - \frac{\tilde{P}}{2}\beta_1\right). \tag{4.22}$$

So we see that if we identify

$$\tilde{P} = P, \quad \tilde{S} = S, \quad \beta_2 = [C], \quad \beta_1 = [C'], \quad n_2 = \chi_h(C) + N_2, \quad n_1 = -\chi_h(C') - N_1 \tag{4.23}$$

with $N = N_1 + N_2$, this almost matches exactly with (4.16), including the correct quantization condition on S.

This match is so good that we expect that generically $[C] \cdot [C'] = 0$. This is certainly true for two generic homology classes in X, and we will assume that for those classes sitting inside a common holomorphic surface Σ it is still generically true. Granted this point, the identifications are interpreted as follows:

At large volume, rank 1 D6-D2-D0 bound states are described by ideal sheaves $\mathcal{I}$ [88, 36, 37, 35] or their duals $\mathcal{I}^*$.[32] More precisely $\mathcal{I}$ corresponds to a collection of curves $C_{\mathcal{I}}$ and points $\pi_{\mathcal{I}}$, where the D2-charge is given by $-\beta + c_2(X)/24$ and the D0-charge by n, where

$$\beta = -\mathrm{ch}_2(\mathcal{I}) = [C_{\mathcal{I}}], \qquad n = \mathrm{ch}_3(\mathcal{I}) = \chi_h(C_{\mathcal{I}} \cup \pi_{\mathcal{I}}) = \chi_h(C_{\mathcal{I}}) + N_{\mathcal{I}}, \tag{4.24}$$

where $N_{\mathcal{I}}$ is the number of points in $\pi_{\mathcal{I}}$ (counted with multiplicities). Taking the dual inverts the odd Chern characters, so the D2-charge of $\mathcal{I}^*$ is given by $-\beta + c_2(X)/24$ where $\beta = [C_{\mathcal{I}}]$ and the D0-charge is $n = -\chi_h(C_{\mathcal{I}}) - N_{\mathcal{I}}$. Hence the above expressions for the charges suggest we identify the Γ_1 system with a D6-D4-D2-D0 bound state described as the dual $\mathcal{I}_1^*$ of an ideal sheaf $\mathcal{I}_1$ "shifted" by a $U(1)$ flux S_1, and Γ_2 similarly as the anti-brane of a D6-D4-D2-D0 bound state described as an ideal sheaf $\mathcal{I}_2$, shifted by S_2. Under this identification, we simply have

$$C_{\mathcal{I}_1} = C', \quad C_{\mathcal{I}_2} = C, \quad N_{\mathcal{I}_1} + N_{\mathcal{I}_2} = N. \tag{4.25}$$

Since the $N_{\mathcal{I}}$ are nonnegative note that n_1 is bounded above, and not below, while n_2 is bounded below, and not above. This will be important in keeping certain signs straight in the derivation of the OSV formula.

Thus we arrive at the following heuristic picture for polar BPS states: The curve collections which are dual to $\gamma + f^{\perp}$ in Σ are the remnants of gases of D2-branes inside D6 and anti-D6 branes with fluxes turned on. The D6-antiD6 condense producing a D4 brane which has captured a gas of D2 and D0 branes.

We can further strengthen this picture by computing moduli degrees of freedom. As explained in section 4.1, a bound state of the two branes under consideration is expected

[32] We do not mean the sheaf-theoretic dual here. If we identify the objects in the category of topological B-branes with the stable objects in the derived category of coherent sheaves then we should take the derived dual. This will be a complex whose cohomology is not supported in a single degree, and hence will not be a sheaf. We thank Paul Aspinwall for pointing this out to us.

to be a cohomology class of a moduli space which is a $\mathbb{CP}^k$ fibration over the product of the moduli spaces of the two constituent branes. Here $k+1$ equals the intersection product of the constituents

$$k+1 = \langle \Gamma_2, \Gamma_1 \rangle = \frac{P^3}{6} + \frac{P \cdot c_2(X)}{12} - P \cdot (\beta_1 + \beta_2) + n_1 - n_2. \tag{4.26}$$

In particular in the case at hand, after freezing the curves and points in the D6 and anti-D6 branes representing the D2-D0 gases, we expect k residual degrees of freedom, coming from light open string modes stretching between the branes.

If the proposed picture is correct, at large radius, these k residual degrees of freedom should correspond to the divisor moduli that remain unfixed in the generic case after requiring the curve collections C and C' and the set of N points p_i to be contained in it. To verify this, rewrite (4.26) using the above identifications as

$$k+1 = \frac{P^3}{6} + \frac{P \cdot c_2(X)}{12} - P \cdot C - \chi_h(C) - P \cdot C' - \chi_h(C') - N \tag{4.27}$$

$$= \int_X e^P \, \mathrm{Td}\, X - \int_C e^P \, \mathrm{Td}\, C - \int_{C'} e^P \, \mathrm{Td}\, C' - N. \tag{4.28}$$

We claim that for P sufficiently large, this agrees exactly with the generic number of deformations of a divisor constrained to contain N points and the curves C and C'. As a simple first check, note that when $C = C' = 0$, $N = 0$, i.e. the pure D4 with at most flux pulled back from $H^2(X)$ turned on, this formula reproduces precisely the dimension $I_P - 1$ of the linear system P. Furthermore, if we think of the divisor for example as a hypersurface given by some homogeneous polynomial equation, then it is clear that if we fix N generic points in X and require the divisor to pass through it, this will give N linear constraints on the polynomial coefficients and thus generically reduce the residual divisor moduli space from $\mathbb{CP}^{I_P-1}$ to $\mathbb{CP}^{I_P-N-1}$.

We now give a proof for the general case for P sufficiently ample. The basic ideas are (i) for P sufficiently ample, we can use index formulas to compute the actual number of deformations, and (ii) the first term in (4.28) is the index counting the number of holomorphic sections of the line bundle describing P, and the second and third terms are the indices counting the number of those sections which when restricted to C resp. C' are nontrivial. Subtracting these terms from the first one thus gives the number of sections of the divisor line bundle which are zero on C and C', i.e. one plus the number of divisor deformations fixing C and C'.

More precisely, this goes as follows.[33] Define the ideal sheaf:

$$0 \to I_C \to \mathcal{O}_X \to \mathcal{O}_C \to 0 \tag{4.29}$$

Our problem is to compute the dimension of $H^0(\mathcal{O}(P) \otimes I_C)$.

Tensor the exact sequence with $\mathcal{O}(P)$. This preserves exact sequences since $\mathcal{O}(P)$ is a line bundle. We write the corresponding long exact sequence

$$0 \to H^0(I_C \otimes \mathcal{O}(P)) \to H^0(\mathcal{O}_X \otimes \mathcal{O}(P)) \to H^0(\mathcal{O}_C \otimes \mathcal{O}(P)) \to H^1(I_C \otimes \mathcal{O}(P)) \to 0 \tag{4.30}$$

[33] We thank E. Diaconescu and T. Pantev for helpful discussoins about this.

For P sufficiently ample $H^1(I_C \otimes \mathcal{O}(P)) = 0$ and moreover $h^1(\mathcal{O}_C(P)) = 0$, but now we can use Riemann-Roch to compute

$$h^0(\mathcal{O}_C(P)) - h^1(\mathcal{O}_C(P)) = \deg(P|_C) - g(C) + 1 \tag{4.31}$$
$$= \int_C e^P \mathrm{Td}(TC) \tag{4.32}$$

Now compare with equation (4.27).

A closely related, but alternative argument proceeds as follows. For concreteness let us take the example of the quintic in $\mathbb{CP}^4$. Let W be the space of homogeneous polynomials in X_0, X_1, X_2, X_3, X_4, and W_d those of degree d. The quintic Calabi-Yau is given by the polynomial equation $Q = 0$, with $Q \in W_5$. Let $\langle Q \rangle$ be the ideal generated by Q and define $W' := W/\langle Q \rangle$, and let W'_d be the restriction of W' to degree d polynomials. Then W'_d (projectivized) can be identified with the moduli space of divisors of degree d on the quintic.

Fix a curve C in the quintic described as the vanishing locus of some homogeneous polynomial ideal $I(C)$ (which includes Q). Then the moduli space of degree d divisors on the quintic which contain C can be identified with the (projectivization of) $I'(C)_d$, the degree d part of $I'(C) := I(C)/\langle Q \rangle$. So we are interested in computing $\dim I'(C)_d$. This is almost directly given by the Hilbert polynomial of C. Define $M(C) := W'/I'(C) = W/I(C)$, i.e. the homogeneous polynomial module associated to C. Then the Hilbert function of C is by definition $f_h(d) := \dim M(C)_d$, and the Hilbert-Serre theorem says that this becomes a polynomial $p_h(d)$ for sufficiently large d. Moreover, that polynomial can be computed from the index theorem.

Since by construction $\dim M(C)_d + \dim I'(C)_d = \dim W'_d$, this gives the expression for $\dim I'(C)_d$:

$$\dim I'(C)_d = \dim W'_d - p_h(d) \tag{4.33}$$

for sufficiently large d. Now we have

$$p_h(d) = \int_C e^{dH} Td(C) = \int_C (1+dH)(1+c_1(C)/2) = dH \cdot C + \chi_h(C) = P \cdot C + \chi_h(C) \tag{4.34}$$

where H is the hyperplane class and $\chi_h(C)$ the holomorphic Euler characteristic of C. Therefore

$$\dim I'(C)_d = I_P - P \cdot C - \chi_h(C) \tag{4.35}$$

in agreement with the above general proof.

These observations give rather strong evidence for the proposed correspondence, although considerably more work would be needed to make things more precise. There is some ambiguity in the identifications in the two pictures, and constructing an exact map between moduli spaces is presumably too much to hope for. In particular we have not analyzed situations in which points or curves coincide so the $\mathbb{CP}^k$ fiber dimension jumps. It appears that here the naive geometrical D4-D2-D0 picture and the D6-anti-D6 bound state picture start to differ, with the latter apparently giving some sort of regularization and stratification of these singular loci. Indeed the considerations of section 5 strongly suggest that in the D6-anti-D6 picture the relevant $\mathbb{CP}^k$ fibrations are always regular for polar states.

We will not attempt to make this map more precise here, but instead proceed by taking the physical D6-anti-D6 picture as a starting point for computing the polar degeneracies. The degree to which the heuristic picture sketched above is accurate will therefore not be essential for the remainder of this paper.

5. Wall-crossing formulae and factorization of indices

In this section we derive wall crossing and factorization formulae for indices, which among other applications will lead to (a refined version of) (3.6) and eventually in section 6 to the factorization $\mathcal{Z}_{\rm top} \sim \mathcal{Z}_{\rm top}\overline{\mathcal{Z}_{\rm top}}$.

5.1 Physical derivation

Let $\mathcal{H}'(\Gamma)_{t_\infty}$ be the (reduced) Hilbert space of BPS states of charge Γ for background moduli t_∞. Then [34]

$$\Omega(\Gamma)|_{t_\infty} := \mathrm{Tr}_{\mathcal{H}'(\Gamma)_{t_\infty}}(-1)^{2J_3'} \tag{5.1}$$

with J_3' the angular momentum with center of mass degrees of freedom factored out.

In the four dimensional supergravity picture, the index (5.1) can get contributions from several distinct multicentered configurations, with different constituent charges summing up to the same total charge Γ, or equivalently from several different topologically distinct attractor flow trees. Apart from the trivial flow tree (i.e. the single flow), all of these will decay when the initial flow tree point t_∞ passes through the wall of marginal stability on which the first split $\Gamma \to \Gamma_1 + \Gamma_2$ of that tree occurs (this will be a different wall for every tree in general). Therefore as soon as there are nontrivial tree contributions to the index, the index can be expected to jump at these walls of marginal stability.

To derive the amount by which the index jumps at a $\Gamma \to \Gamma_1 + \Gamma_2$ MS wall, we will first assume Γ_1 and Γ_2 are both primitive. In that case all states decaying at this wall will necessarily look like two clusters of bound particles of charge Γ_1 resp. Γ_2, which get infinitely far separated from each other when the wall is approached (recall eq. (3.22)). Denote the part of $\mathcal{H}'(\Gamma)_{t_\infty}$ corresponding to these nearly decaying states by $\mathcal{H}'(\Gamma \to \Gamma_1 + \Gamma_2)_{t_\infty}$, where we let $t_\infty \to t_{\rm ms}$, $t_{\rm ms}$ being a point on the marginal stability wall under consideration. One expects this Hilbert space to factorize as

$$\mathcal{H}'(\Gamma \to \Gamma_1 + \Gamma_2)_{t_{\rm ms}} = \left(\frac{|\mathbf{I_{12}}|-\mathbf{1}}{\mathbf{2}}\right) \otimes \mathcal{H}'(\Gamma_1)_{t_{\rm ms}} \otimes \mathcal{H}'(\Gamma_2)_{t_{\rm ms}}. \tag{5.2}$$

The first factor comes from the quantization of the centers of mass of the two clusters and their associated fermionic degrees of freedom, which as reviewed in section 3.2 yields a spin $J_3' = \frac{|I_{12}|-1}{2}$ multiplet, where $I_{12} \equiv \langle \Gamma_1, \Gamma_2 \rangle$. Thus, one expects a jump in the index given by

$$\Delta\Omega|_{t_{\rm ms}} = (-1)^{I_{12}-1}\,|I_{12}|\,\Omega(\Gamma_1)|_{t_{\rm ms}}\,\Omega(\Gamma_2)|_{t_{\rm ms}} \tag{5.3}$$

[34] Elsewhere in the paper we also use the notation $\mathcal{H}'(\Gamma; t_\infty)$ and $\Omega(\Gamma; t_\infty)$.

when going from the unstable to the stable side of the marginal stability wall.[35]

The main physical input that went into this derivation is the factorization of Hilbert spaces (5.2) for infinitely separated clusters. Although plausible, this is not completely obvious, since one could imagine interactions e.g. between the spin of one cluster and the magnetic field produced by the other cluster, which could spoil supersymmetry by a tiny but nonzero bit. The spin of the clusters depends on the relative positions of the centers (see (3.24)), so translated to these degrees of freedom one should check if there are non-infinitesimal effects on the relative BPS position constraints of the centers within one cluster, coming from the presence of the second cluster. If the integrability constraints admit solutions with a cluster of centers $\vec{x}_\alpha$ going to infinity, while other centers $\vec{x}'_i$ remain finite then clearly the effect on the remaining centers $\vec{x}'_i$ in (3.21) is negligible and amounts effectively merely to an infinitesimal shift of the constant term on the right hand side of the constraint equations. (The cases where this constant term is zero are nongeneric and can be eliminated by slightly perturbing t_∞, which for the sake of this argument we are free to choose anywhere as long as t_∞ stays very near the wall of marginal stability on the stable side). To strengthen our confidence in these arguments, we will give several mathematical tests of the wall crossing formula in the following subsections.

The wall crossing formula can be used to derive a refined version of (3.6). Fix some $t_\infty = t_i$ and consider all splits $\Gamma \to \Gamma_1 + \Gamma_2$ encountered along the single Γ attractor flow starting at $t = t_i$ and ending at $t = t_f$, where t_f is either the attractor point or a zero of $Z(\Gamma)$. Note that by the time this endpoint is reached, all configurations contributing to the index $\Omega(\Gamma; t_\infty)$ that could decay, have decayed; there are no nontrivial trees left at this point. Repeating the wall crossing formula (5.3) for each jump encountered along the attractor flow gives the formula

$$\Omega(\Gamma)|_{t_i} = \Omega(\Gamma)|_{t_f} + \sum_{\Gamma \to \Gamma_1 + \Gamma_2} (-1)^{\langle \Gamma_1, \Gamma_2 \rangle - 1} |\langle \Gamma_1, \Gamma_2 \rangle| \; \Omega(\Gamma_1)|_{t_{\rm ms}(\Gamma_1, \Gamma_2, t_i)} \; \Omega(\Gamma_2)|_{t_{\rm ms}(\Gamma_1, \Gamma_2, t_i)} \tag{5.4}$$

where the sum is over all $\Gamma \to \Gamma_1 + \Gamma_2$ splittings along the attractor flow and $t_{\rm ms}(\Gamma_1, \Gamma_2, t_i)$ is the point where the flow crosses the corresponding $\Gamma \to \Gamma_1 + \Gamma_2$ marginal stability wall. When the final point corresponds to a zero, as is the case for polar D4-D2-D0 states, we moreover have $\Omega(\Gamma)|_{t_f} = 0$, and all contributions have a factorized form.

Iteratively repeating this for each of the $\Omega(\Gamma_i)|_{t_{\rm ms}}$ eventually gives the expression

$$\Omega(\Gamma)|_{t_\infty} = \sum_{T \in \mathcal{T}(\Gamma, t_\infty)} \prod_{\Gamma_a \to \Gamma_b + \Gamma_c \in {\rm Vert}(T)} (-1)^{\langle \Gamma_b, \Gamma_c \rangle - 1} |\langle \Gamma_b, \Gamma_c \rangle| \prod_{\Gamma_i \in {\rm Term}(T)} \Omega(\Gamma_i, t_*(\Gamma_i)) \tag{5.5}$$

where $\mathcal{T}(\Gamma, t_\infty)$ is the set of all attractor flow trees of total charge Γ starting at t_∞, ${\rm Vert}(T)$ is the set of vertices of the flow tree T, characterized as splits $\Gamma_a \to \Gamma_b + \Gamma_c$, ${\rm Term}(T)$ is the set of terminal charges of the flow tree T, and $t_*(\Gamma_i)$ the attractor point of Γ_i. Thus

[35] Of course, (5.2) implies something stronger than (5.3). We could for example state an analogous wall-crossing formula for the full character ${\rm Tr} y^{2J'_3}$ implying a wall-crossing formula for the Hodge polynomials of the relevant moduli spaces.

we see that flow trees give a canonical way of reducing indices in general backgrounds to irreducible[36] indices $\Omega(\Gamma_i, t_*(\Gamma_i))$ associated to black holes or simple particles.

In the wall crossing formula (5.3) and the subsequent formulae we have assumed that all splits $\Gamma \to \Gamma_1 + \Gamma_2$ are primitive, i.e. no integral Γ_1' and integer $N_1 > 1$ exist such that $\Gamma_1 = N_1 \Gamma_1'$, and similarly for Γ_2. In general this need not be the case. It is possible to extend the wall crossing formula (5.3) to some nonprimitive cases as well. This is done most efficiently by using generating functions; examples will be analyzed in detail in the section 6.1, but for completeness we give a more general wall crossing formula here already, for arbitrary splits $\Gamma \to \Gamma_1 + N\Gamma_2$, $N \in \mathbb{Z}^+$ (which is a nonprimitive split when $N > 1$):

$$\Omega(\Gamma_1)|_{t_{\rm ms}} + \sum_{N>0} \Delta\Omega(\Gamma_1 + N\Gamma_2)|_{t_{\rm ms}}\, q^N = \Omega(\Gamma_1)|_{t_{\rm ms}} \prod_{k>0} \Big(1 - (-1)^{k\langle \Gamma_1, \Gamma_2 \rangle}\, q^k\Big)^{k|\langle \Gamma_1, \Gamma_2 \rangle|\, \Omega(k\Gamma_2)|_{t_{\rm ms}}} \tag{5.6}$$

where $\Delta\Omega$ denotes the index jump at the appropriate marginal stability point $t_{\rm ms}(\Gamma_1, \Gamma_2, t_\infty)$, going from unstable to stable side. These splits correspond to "halo" states, consisting of N Γ_2 particles moving on a sphere around Γ_1. We will see several special cases in section 6.1, after which it will be clear that this formula is the correct generalization. Note that it reduces to (5.3) for $N = 1$. It would be interesting to generalize this formula further to splits $\Gamma \to N_1\Gamma_1 + N_2\Gamma_2$, but we expect this to be significantly more complicated. In this case the configuration space will be much more complicated. Moreover, from our arguments above $\Delta\Omega(N_1\Gamma_1 + N_2\Gamma_2)$ is related to the Euler character of a quiver with 2 nodes with dimension vector (N_1, N_2) and $k = \langle \Gamma_1, \Gamma_2 \rangle$ arrows. However, the known expressions for these Euler characters are very complicated [93].

We will however primarily use (5.4) to factorize the polar part of the D4-partition function, and will argue that for the purpose of deriving the OSV conjecture it is sufficient to restrict to the contribution from splits in two clusters with a single D6 and a single anti-D6 brane charge, which are of course automatically primitive.

Finally, note that we could have tried to derive the wall crossing formula microscopically from (4.5) (adding the proper signs obtained from the identification of J_3' with Lefshetz spin as explained above (2.7)). However, at least to be able to use this in a straightforward fashion, this would have required us to assume that whenever a D-brane is close to decaying into two branes, its connected moduli space component $\mathcal{M}$ has the structure of a $\mathbb{CP}^{|\langle \Gamma_1, \Gamma_2 \rangle| - 1}$ fibration over the product of the moduli spaces $\mathcal{M}_1$ and $\mathcal{M}_2$ of the consitutent branes, *without* any degenerations of the fiber. The latter is not clear *a priori*. Turning things around, the physical arguments given above give a prediction that the fibration will indeed be regular in these cases, or at least that this can be effectively assumed for the purpose of computing the jump of the index.

In the following we will test our wall-crossing formula microscopically, and we will see that indeed this regular fibration structure arises in association to decaying states, often in rather nontrivial ways.

[36] irreducible in the sense that they correspond to states which cannot be made to decay — there might be more refined reductions which further factorize even these irreducible indices.

5.2 Mathematical tests and applications

We have verified our wall crossing formulae, and the index factorizations derived from it, both microscopically by comparing to large radius geometrical results as well as by studying examples of quiver moduli spaces. The latter often are under good mathematical control [93], and their relation to multicentered configurations is well understood in a number of cases [65].

The simplest example is a pure (very ample) D4 of charge P, which as we saw in the previous sections corresponds to a bound state of a D6 and an anti-D6 with suitable fluxes turned on, with corresponding charges $\Gamma_1 = e^{S_1}(1 + c_2/24)$, $\Gamma_2 = -e^{S_2}(1 + c_2/24)$ such that $P = S_1 - S_2$. The intersection product between the two constituents equals $I_P = P^3/6 + c_2 \cdot P/12$. For a single D6 brane with flux the attractor point lies on the boundary of moduli space. However, the low energy gauge theory is free Maxwell theory and hence, on a Calabi-Yau X with proper $SU(3)$ holonomy, there is a unique ground state of the Maxwell theory in a fixed flux sector. [37] Therefore, $\mathcal{H}'(\Gamma; t_*(\Gamma))$ is one-dimensional, and hence $\Omega(\Gamma_i; t_*(\Gamma_i)) = 1$. We will assume $\Gamma \to \Gamma_1 + \Gamma_2$ is the only flow tree (an assertion very well supported by our numerical and analytical searches), and hence equation (5.5) immediately gives $\Omega(P; t_\infty) = (-1)^{I_P - 1} I_P$. This is in exact agreement with the microscopic index computed as the euler characteristic of the linear system corresponding to the divisor P, which is $\mathbb{CP}^{I_P - 1}$. Note that this is also the moduli space of a two-node quiver with I_P arrows and dimension vector $(1, 1)$, in accordance with the discussion in section 4.2, and in particular (4.8).

Further tests along these lines can be extracted from the discussion towards the end of section 4.3.

In the following subsections we will consider a number of more complicated examples, some of which are of independent interest. Finally, the results for halo degeneracies we will obtain in the next sections can also be considered as further tests of these ideas.

5.2.1 Four node quiver without closed loops

To verify our physical arguments for the absence of long distance spin-spin interactions spoiling factorization, we consider a system described by the quiver of fig. 11, close to a locus where the two nodes on the left hand side split off from those on the right hand side. More precisely we will go to a locus of FI parameter space where the multicentered solutions of (4.9) split in these two clusters and we thus expect the index to factorize accordingly. We test this by showing that the quiver moduli space (4.8) is a $\mathbb{CP}^{I-1}$ fibration over the product of the moduli spaces of the two subsectors, with I the intersection product between the two clusters, and that its cohomology factorizes as physically expected. As discussed at

[37] In this paper we have ignored torsion in the cohomology groups. However, at this point torsion in $H^2(X; \mathbb{Z})$ plays an interesting role. In this case one cannot simultaneously specify the electric and magnetic flux sectors on the D6 brane, and in fact, the ground states of the theory form a representation of the Heisenberg group extension of $H^2_{tors}(X; \mathbb{Z}) \times H^5_{tors}(X; \mathbb{Z})$ defined by the torsion pairing [120, 121]. Thus, it is more appropriate to take $\Omega(\Gamma_i; t_*(\Gamma_i)) = |H^2_{tors}(X; \mathbb{Z})|$. Since different attractor flow trees terminate on different numbers of pure six branes the torsion effects will modify the indices in interesting ways. We have not systematically investigated these consequences of nonzero torsion.

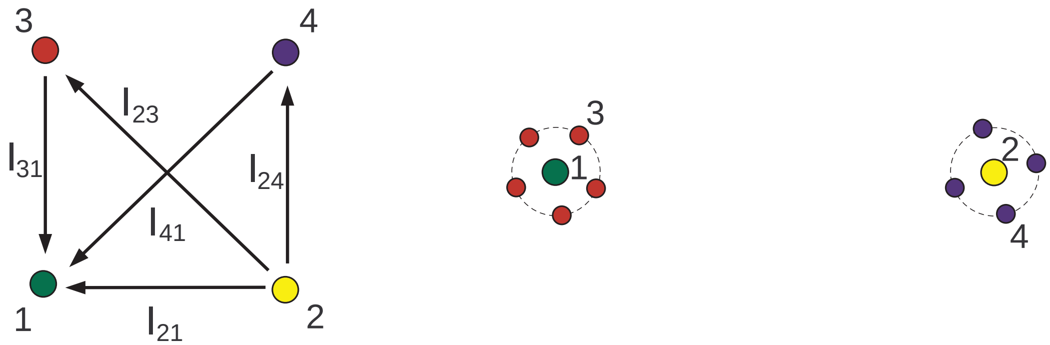


Figure 11: Left: Four node quiver without closed loops. The indicated arrows are I_{ij}-fold degenerate, with $I_{ij} \geq 0$. The dimension vector is taken to be $(1, 1, d_3, d_4)$, with $d_3, d_4 \geq 1$. **Right:** Corresponding splitting multicentered configuration when $\vartheta_1 + d_3\vartheta_3$ is approaching zero.

the end of section 5.1, this is sufficient to reproduce the wall crossing formula for the index. However establishing this regular fibration structure turns out to be rather nontrivial.

We will not try to give an actual physical or flow tree realization of this quiver, which is not necessary for the kind of comparison we are trying to make here.

The morphisms (or bosonic stretched open string modes) are described by

$$\phi_{21} \in \ \mathbb{C}^{I_{21}} \tag{5.7}$$
$$\phi_{31} \in \ \mathrm{Mat}(1, d_3) \times \mathbb{C}^{I_{31}} \tag{5.8}$$
$$\phi_{23} \in \ \mathrm{Mat}(d_3, 1) \times \mathbb{C}^{I_{23}} \tag{5.9}$$
$$\phi_{41} \in \ \mathrm{Mat}(1, d_4) \times \mathbb{C}^{I_{41}} \tag{5.10}$$
$$\phi_{24} \in \ \mathrm{Mat}(d_4, 1) \times \mathbb{C}^{I_{24}} \tag{5.11}$$

denote $\phi_{31}^{\alpha,j}$ with $\alpha = 1, \ldots, d_3$, $j = 1, \ldots, I_{31}$ and $\phi_{23}^{j,\alpha}$ with $j = 1, \ldots, I_{23}$, and similarly for ϕ_{41}, ϕ_{24}. The D-term equations are given by (4.6):

$$-|\phi_{21}|^2 - |\phi_{31}|^2 - |\phi_{41}|^2 = \vartheta_1 \tag{5.12}$$
$$|\phi_{21}|^2 + |\phi_{23}|^2 + |\phi_{24}|^2 = \vartheta_2 \tag{5.13}$$
$$\sum_{j=1}^{I_{31}} \phi_{31}^{\alpha,j} (\phi_{31}^{\beta,j})^* - \sum_{j=1}^{I_{23}} (\phi_{23}^{j,\alpha})^* \phi_{23}^{j,\beta} = \vartheta_3 \delta^{\alpha,\beta} \tag{5.14}$$
$$\sum_{j=1}^{I_{41}} \phi_{41}^{\alpha,j} (\phi_{41}^{\beta,j})^* - \sum_{j=1}^{I_{24}} (\phi_{24}^{j,\alpha})^* \phi_{24}^{j,\beta} = \vartheta_4 \delta^{\alpha,\beta} \tag{5.15}$$

with ϑ_i as in (4.7) and as usual $\vartheta_1 + \vartheta_2 + d_3\vartheta_3 + d_4\vartheta_4 = 0$. The corresponding supersymmetric particle configuration constraints (4.9) are

$$-\frac{I_{21}}{|\vec{x}_1 - \vec{x}_2|} - \sum_{\beta=1}^{d_3} \frac{I_{31}}{|\vec{x}_1 - \vec{x}_{3\beta}|} - \sum_{\beta=1}^{d_4} \frac{I_{41}}{|\vec{x}_1 - \vec{x}_{4\beta}|} = \vartheta_1 \tag{5.16}$$

$$\frac{I_{21}}{|\vec{x}_2-\vec{x}_1|}+\sum_{\beta=1}^{d_3}\frac{I_{23}}{|\vec{x}_2-\vec{x}_{3\beta}|}+\sum_{\beta=1}^{d_4}\frac{I_{24}}{|\vec{x}_2-\vec{x}_{4\beta}|}=\vartheta_2 \tag{5.17}$$

$$\frac{I_{31}}{|\vec{x}_{3\alpha}-\vec{x}_1|}-\frac{I_{23}}{|\vec{x}_3-\vec{x}_{2\alpha}|}=\vartheta_3 \tag{5.18}$$

$$\frac{I_{41}}{|\vec{x}_{4\alpha}-\vec{x}_1|}-\frac{I_{24}}{|\vec{x}_{4\alpha}-\vec{x}_2|}=\vartheta_4, \tag{5.19}$$

which as mentioned in section (4.2) coincides with the supergravity position constraints (4.10) in the regime of validity of the quiver quantum mechanics, $\theta_i \ll 1$, since $\theta_i \approx \vartheta_i$ in this regime.

Now, we want $\vec{x}_1, \vec{x}_{3,\alpha} \to \infty$ with $|\vec{x}_1-\vec{x}_{3,\alpha}|$ held finite. Therefore, in the quiver picture we should send $\vartheta_1+d_3\vartheta_3 \to 0$, holding $\vartheta_1, \vartheta_3, \vartheta_2, \vartheta_4$ all nonzero. The system then splits in two clusters with charges $\Gamma_1+d_3\Gamma_3$ and $\Gamma_2+d_4\Gamma_4$, respectively, with mutual intersection product

$$I=\langle\Gamma_2+d_4\Gamma_4,\Gamma_1+d_3\Gamma_3\rangle=I_{21}+d_3I_{23}+d_4I_{41}. \tag{5.20}$$

Clearly we must have $\vartheta_1<0$, and therefore $\vartheta_3>0$. Similarly, since $\vartheta_1+d_3\vartheta_3=-(\vartheta_2+d_4\vartheta_4)$ and since $\vartheta_2>0$ we must have $\vartheta_4<0$.

It follows from the third and fourth D-term equations that there is a well-defined projection to a product of Grassmannians: $[\phi_{31}^{\alpha j}] \in Gr(d_3, I_{31})$ and $[\phi_{24}^{j\alpha}] \in Gr(d_4, I_{24})$. This leaves ϕ_{23} and ϕ_{41} undetermined, and the remaining equations determines the fiber of the map to be a complex projective space so that the moduli space is a smooth fibration [38]

$$\mathbb{C}P^{I_{21}+d_3I_{23}+d_4I_{41}-1} \to \mathcal{M} \to Gr(d_3,I_{31})\times Gr(d_4,I_{24}). \tag{5.21}$$

Note that the Grassmannians are the moduli spaces $\mathcal{M}_1$, $\mathcal{M}_2$ of the two 2-node sub-quivers in which our 4-node quiver splits. Restricting the gauge invariant form $d\phi_{21}\wedge d\overline{\phi_{21}}+d\phi_{23}\wedge d\overline{\phi_{23}}+d\phi_{41}\wedge d\overline{\phi_{41}}$ to the fibers gives a generator of the cohomology of the fibers, so by the Leray-Hirsch theorem the cohomology factorizes:

$$H^*(\mathcal{M})=H^*(\mathbb{C}P^{I_{21}+d_3I_{23}+d_4I_{41}-1})\otimes H^*(\mathcal{M}_1)\otimes H^*(\mathcal{M}_2) \tag{5.22}$$

Comparing with (5.20) we see that the factorization (5.22) is precisely that predicted by the physics, and rather nontrivially so.

5.2.2 A D6-D2-D0 as a 3 centered $D6-D6-\overline{D6}$ bound state

Next we give a 3-centered example with an actual flow tree realization and a microscopic description as a geometric D-brane in the IIA large radius limit. Consider a bound state of the following three charges

$$\Gamma_1=e^U(1+\frac{c_2}{24}),\quad \Gamma_2=e^V(1+\frac{c_2}{24}),\quad \Gamma_3=-e^{U+V}(1+\frac{c_2}{24}) \tag{5.23}$$

with $U, V, V-U$ positive divisors (i.e. inside the Kähler cone). The total charge is

$$\Gamma=1\,-\,UV+\frac{c_2}{24}\,-\,\frac{1}{2}(UV^2+U^2V) \tag{5.24}$$

[38] We omit many details in the argument here.

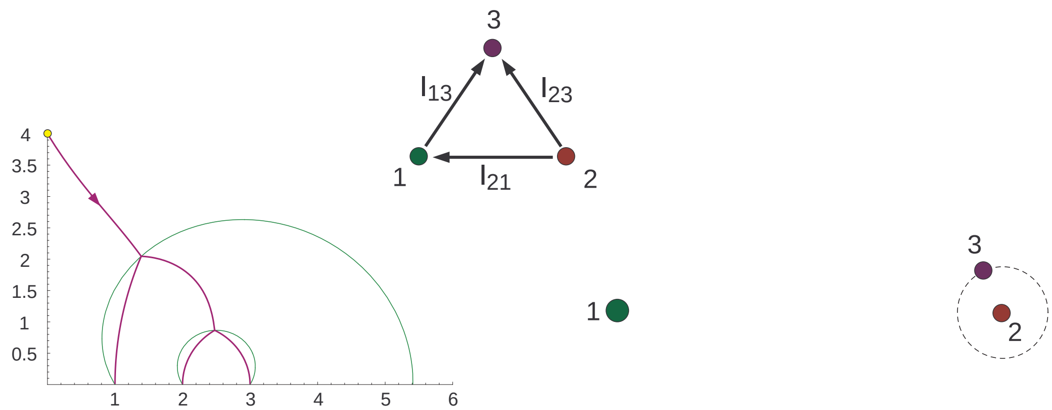


Figure 12: Left: Flow tree corresponding to the D6-D2-D0 system described in the text, in the 1-modulus case with $U = D_1$, $V = 2\,D_1$. The attractor points of $\Gamma_1, \Gamma_2, \Gamma_3$ are at $x = 1, 2, 3$ respectively. **Up:** Corresponding quiver formally associated to this system, but in fact *not* describing the system. **Right:** Corresponding splitting multicentered Sun-Earth-Moon type configuration when the MS wall $\arg Z_1 = \arg(Z_2 + Z_3)$ is approached.

so this is a D6-D2-D0 bound state. Denoting $I_{ij} \equiv \langle \Gamma_i, \Gamma_j \rangle$, we have

$$I_{21} = \frac{(V-U)^3}{6} + \frac{c_2 \cdot (V-U)}{12}, \qquad I_{23} = \frac{U^3}{6} + \frac{c_2 \cdot U}{12}, \qquad I_{13} = \frac{V^3}{6} + \frac{c_2 \cdot V}{12}. \tag{5.25}$$

Because U, V and $V-U$ are all positive, all of these intersection numbers are positive. An example of a corresponding attractor flow tree in the one modulus case is shown in fig. 12, as well as the quiver encoding the intersection products and the corresponding multicentered configuration when approaching a $(1, 2+3)$ line of marginal stability. Crucial is that the sequence of splits is $(123) \to (1,23) \to (1,2,3)$. At least in the one modulus case, it can be checked that this is the only possible flow tree for the given charges starting from large $\operatorname{Im} t$. Let us assume this is true in the general case as well.

Our factorization arguments immediately yield the following index of BPS states associated to this flow tree:

$$\Omega = (-1)^{I_{21}+I_{23}+I_{13}}\, |I_{13} - I_{21}|\, I_{23} = (-1)^{I_U + I_V + I_{V-U}} |I_V - I_{V-U}| I_U, \tag{5.26}$$

where we used the notation $I_U \equiv \frac{U^3}{6} + \frac{c_2 \cdot U}{12}$ and so on, and the fact that for proper $SU(3)$ holonomy Calabi-Yau manifolds, the pure D6 has a unique ground state after factoring out the center of mass hypermultiplet, i.e. $\Omega(e^S\,\Gamma(0,0))|_t = 1$.

Now let us compare this to the microscopic large radius geometrical picture of this D6-D2-D0 as the ideal sheaf $\mathcal{I}_C$ given by the curve $C = U \cap V$. The charges are given by (4.24), which yields, using the adjunction formula,

$$q_{D2} = -U \cdot V + \frac{c_2}{24}, \qquad q_0 = \chi_h(C) = -\frac{1}{2}\left(C^2|_V + C \cdot V\right) = -\frac{1}{2}(U^2 V + U V^2), \tag{5.27}$$

in agreement with (5.24). We can parametrize the moduli space of this ideal sheaf as follows. First recall that the moduli space of very ample divisors D ($= U$, V and $V-U$

here) is parametrized by the vector space of holomorphic sections s_D of the associated line bundles $\mathcal{L}_D$, modulo overall rescaling of the section. That is,

$$\mathcal{M}_D = \mathbb{CP}^{I_D-1}. \tag{5.28}$$

Now pick a divisor representative U_0 in the class U, described by the vanishing locus of a section s_{U_0} of $\mathcal{L}_U$. Note that we can write any section of $\mathcal{L}_V$ as

$$s_V = s_{U_0}\, s_{V-U} + \tilde{s}_V, \tag{5.29}$$

where s_{V-U} is some section of $\mathcal{L}_{V-U}$ and $\tilde{s}_V$ a section of $\mathcal{L}_V$. Conversely, any such expression gives a holomorphic section of $\mathcal{L}_V$. Now changing s_{V-U} in this expression will not change $C_0 := U_0 \cap V = \{s_{U_0} = 0\} \cap \{s_V = 0\}$, and thus the moduli space of curves C_0 is described by the vector space of sections $\tilde{s}_V$ of $\mathcal{L}_V$ modulo products of sections of $\mathcal{L}_{V-U}$ with s_{U_0}, and modulo overall rescalings, that is

$$\mathcal{M}_{C_0} = \mathbb{CP}^{I_V - I_{V-U} - 1}. \tag{5.30}$$

In conclusion, the sheaf moduli space $\mathcal{M}_C$ is a regular $\mathbb{CP}^{I_V-I_{V-U}-1}$ fibration over $\mathbb{CP}^{I_U-1}$. Computing the Euler characteristic of this space immediately reproduces (5.26).

Note that for this example, there is no region in moduli space where all phases of the constituents line up, as can be seen directly in fig. 12 since the two marginal stability lines do not intersect. As a result, the quiver quantum mechanics picture as reviewed in section 4.2 is not reliable in this case. And indeed, if one tries to compute the index as the euler characteristic of the moduli space $\mathcal{M}$ for the quiver of fig. 12 with dimension vector $(1,1,1)$, as given by (4.8) (with $W = 0$ since there is no closed loop), one finds the *wrong* result $\Omega = (-1)^{I_{21}+I_{23}+I_{13}}\,(I_{21}+I_{23})\,I_{13}$ or $(-1)^{I_{21}+I_{23}+I_{13}}\,(I_{13}+I_{23})\,I_{21}$, depending on the sign of ϑ_1. (This can be computed with the methods described in section 5.2.1.) This illustrates that the split flow picture is more general than the quiver picture.

It can be shown that the correct index is that of the part of the quiver BPS Hilbert space which jumps at the MS wall $\vartheta_1 = 0$, i.e. the difference of the Hilbert spaces for $\vartheta_1 > 0$ and $\vartheta_1 < 0$, suggesting an identification of the ideal sheaf cohomology with this part of the quiver cohomology. It would be interesting to clarify this point further.

When we invert $U \to -U$ in (5.23) the situation changes significantly. In this case, there *is* a region of moduli space where all phases line up, and the quiver description becomes accurate. The relevant quiver now has a closed loop however, allowing a nontrivial superpotential. We now turn to this case.

5.2.3 Three node quiver with closed loop

Our third nontrivial example is given by the quiver given in fig. 6*b*, which has a closed loop. The latter brings in some qualitatively new features, such as the presence of a superpotential and the possibility of scaling solutions.

For simplicity we take the dimension vector to be $(1,1,1)$, and assume no internal moduli associated to the vertices. This could correspond for instance to a bound state of

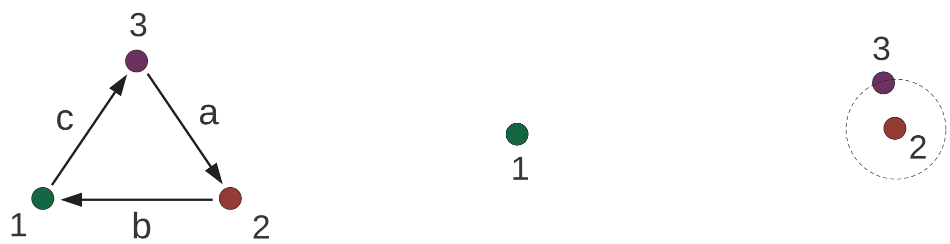


Figure 13: Left: Three node quiver with closed loop. **Right:** Corresponding splitting multicentered configuration when $b > a + c$ and θ_1 approaches zero.

three single D6 or anti-D6 branes with suitable $U(1)$ fluxes turned on on their worldvolumes. In particular our previous example (5.23) with U inverted, i.e.,

$$\Gamma_1 = e^{-U}, \quad \Gamma_2 = e^{V}, \quad \Gamma_3 = -e^{V-U} \tag{5.31}$$

realizes this, where we take U, V to be positive divisors and we are dropping c_2 corrections for simplicity. We have $a = U^3/6$, $b = (U+V)^3/6$, $c = V^3/6$. Since U, V are positive divisors we have $b > a + c$.

Fig. 14 shows a one modulus example of flows associated with (5.31) with $U = D_1$, $V = 2D_1$ and three different initial points. No flow trees exist in the large radius regime. In other regions, one or more of the three possible tree topologies $(1,(2,3))$, $(2,(3,1))$, $(3,(1,2))$ is realized. For initial point A, we have only $(1,(2,3))$. When moving to point B, we pass through the marginal stability line where Z_3 and $Z_1 + Z_2$ line up, i.e. $\theta_3 = 0$. The $(1,(2,3))$ remains alive $(B1)$, but a new tree, of topology $(3,(1,2))$ $(B2)$ comes into existence. So in this case the total Hilbert space $\mathcal{H}(\Gamma; t_\infty)$ will be partitioned by two trees. Finally, when moving to point C, we do not pass through any relevant marginal stability line, but along the way a tree-topology-changing transition takes place: at some point, both flow trees we had in B become degenerate (with one 4-valent vertex instead of two 3-valent) and identical, and going beyond that, we are left with one new tree, of topology $(2,(3,1))$.

According to our general framework, the index should jump between A and B, but not between B and C. This is confirmed by the explicit expressions obtained from our factorization formulae

$$\begin{aligned}
\Omega(A) &= (-1)^{\langle\Gamma_1,\Gamma_2+\Gamma_3\rangle+\langle\Gamma_2,\Gamma_3\rangle}\,|\langle\Gamma_1,\Gamma_2+\Gamma_3\rangle||\langle\Gamma_2,\Gamma_3\rangle| = (-1)^{a+b+c}(b-c)a && (5.32)\\
\Omega(B) &= \Omega(B1) + \Omega(B2) = (-1)^{a+b+c}\,((b-c)a + (c-a)b) && (5.33)\\
&= (-1)^{a+b+c}(b-a)c && (5.34)\\
\Omega(C) &= (-1)^{a+b+c}(b-a)c, && (5.35)
\end{aligned}$$

where we used $b > a + c$ and $c > a$. Note that the two B-trees indeed nicely combine to give the same index of the single C-tree!

Let us now turn to the microscopic description. Since there are no flow trees starting from the large radius regime, there won't be a geometrical large radius D-brane realization.

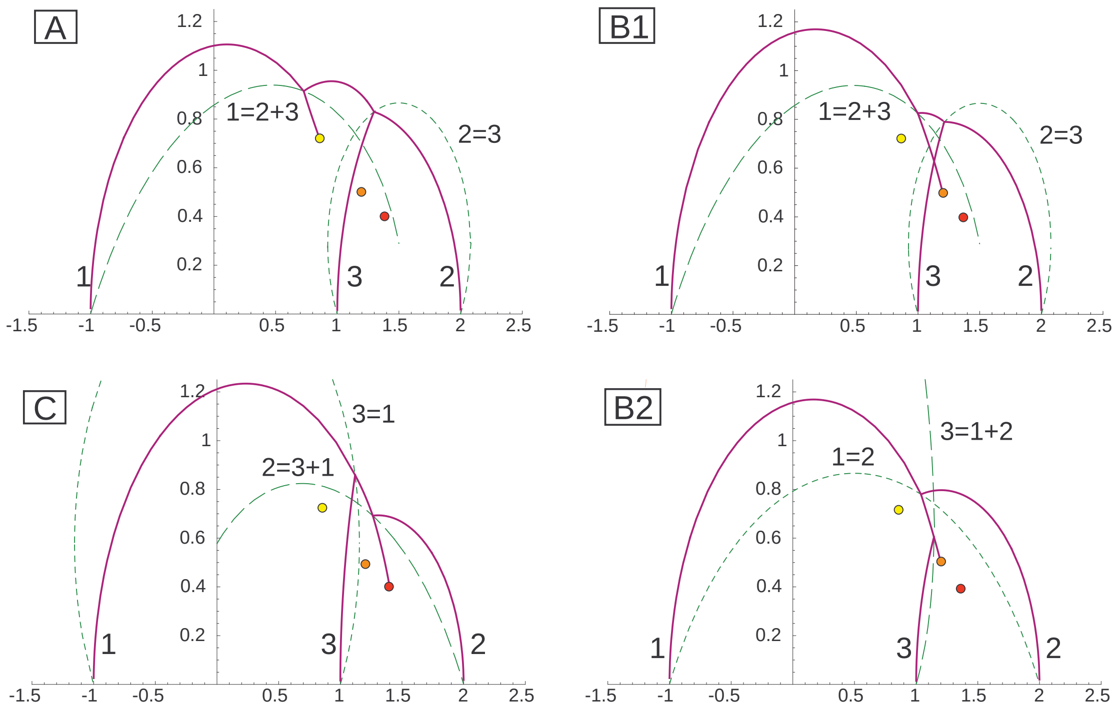


Figure 14: One modulus example of a realization of the quiver in fig. 13. The three terminal charges Γ_i are given by (5.31) (dropping c_2 corrections) with $U = D_1$, $V = 2D_2$, and the corresponding terminal flows labeled by 1, 2 and 3. The green dotted lines are lines of marginal stability. The ms line "$2 = 3$" corresponds to Z_2 and Z_3 lining up, "$1 = 2 + 3$" to Z_1 and $Z_2 + Z_3$ lining up, and so on. For each flow tree, we only show the ms lines on which the tree has vertices; this is different for each tree. Three different initial points are considered, corresponding to the labels A, B and C. We consider the three kinds of flow patterns (1(23)), (2(13)) and (3(12)) at each of the initial points. The initial (yellow) point in case A only supports the flow (1(23)). The initial point in case B supports the two flows (1(23)) and (3(12)), illustrated in $B1$ and $B2$. Finally, the initial point in case C again supports only the flow (2(13)). There are also regions where no flow tree exists, for example the large $\mathrm{Im}\, t$ region. Here the attractor flow is a single centered flow for the total charge which crashes on a zero of $Z(\Gamma; t)$.

Indeed, we now have a D6-D2-D0 charge with *positive* D2 charge, which never exists as a BPS state in the large radius regime. Fortunately however, we see that there is a region in moduli space where all phases of the nodes line up, so we can use quiver quantum mechanics to verify our results.

As in section 5.2.1, we can actually verify our results independently of the split flow picture by just comparing the results obtained from the 3-particle quantum mechanics to those from the microscopic quiver moduli space.

We label the stretched open string scalars by $z_i, i = 1, \ldots, I_{13} := c$, $x_j, j = 1 \ldots, I_{32} := a$, $y_k, k = 1, \ldots, I_{21} := b$. The D-term constraints are

$$\sum_i |z_i|^2 - \sum_k |y_k|^2 = \vartheta_1 \tag{5.36}$$

$$\sum_k |y_k|^2 - \sum_j |x_j|^2 = \vartheta_2 \tag{5.37}$$

$$\sum_j |x_j|^2 - \sum_i |z_i|^2 = \vartheta_3, \tag{5.38}$$

Since the quiver has a closed loop, a nontrivial gauge invariant superpotential is possible. We assume this to have a generic cubic form

$$W(x,y,z) = \sum_{ijk} c_{ijk} z_i x_j y_k. \tag{5.39}$$

Higher order terms can self-consistently be neglected as long as the (x,y,z) are small. As we will see, this can be enforced by making $|\vartheta_i|$ sufficiently small.

From the discussion in section 3.8 we expect factorization when at least one of the triangle inequalities (3.59) is violated, say, as in our concrete realization described above,

$$b > a + c. \tag{5.40}$$

As we saw in section 3.8, for $\theta_3 > 0$, the configuration will split for θ_1 approaching 0 (from below) by separating $\vec{x}_1$ infinitely far from $\vec{x}_2$ and $\vec{x}_3$, so our general physical arguments lead to an index for this particular system given by

$$\Omega_{\rm macro} = (-1)^{c-b-1}\,(b-c)\,\Omega(1)\,\Omega(2+3) = (-1)^{c+b+a}\,(b-c)\,a. \tag{5.41}$$

This corresponds to our case A.

From the point of view of the microscopic quiver moduli space

$$\mathcal{M} := \{(x,y,z)|(5.36)-(5.38) \text{ satisfied and } \partial W = 0\}/U(1)^3 \tag{5.42}$$

the result (5.41) is not at all obvious. For instance it appears rather mysterious why the microscopic quiver description should care about triangle inequalities.

Let us therefore compute the index directly from the quiver moduli space, and check if factorization holds when expected. The index of this system is given by

$$\Omega_{\rm micro} = (-1)^{\dim \mathcal{M}} \chi(\mathcal{M}) = (-1)^{c+a+b}\,\chi(\mathcal{M}). \tag{5.43}$$

(Recall the origin of the sign factor is the identification of J_3' with Lefshetz spin, as explained above (2.7).) For generic c_{ijk}, the solutions to $\partial W = 0$ split in three branches, one with $x = 0$, one with $y = 0$ and one with $z = 0$. This can be seen as follows.[39] Assume there are other solutions, i.e. with $x \neq 0$, $y \neq 0$, $z \neq 0$. Relabeling indices we can assume say $x_1 \neq 0$, $y_1 \neq 0$, $z_1 \neq 0$. Now note that the equations have scaling symmetries $x \to \lambda_1 x$, $y \to \lambda_2 y$, $z \to \lambda_3 z$, so without loss of generality we can assume $x_1 = y_1 = z_1 = 1$. The equations corresponding to partial derivatives with respect to the other variables are then a nice set S of equations that can be solved with a finite set of solutions. The equations corresponding to partial derivatives with respect to x_1, y_1, z_1 are an extra set of constraints. From the

[39] We thank Davide Gaiotto for providing this argument.

homogeneity of $W(x,y,z)$ it follows that these three extra equations are satisfied iff the superpotential evaluated on the solutions to S is zero. On the other hand the coefficient c_{111} does not enter the first set of equations S. Picking different values for c_{111} one can get any possible value for the superpotential. It follows that branches with x, y, z all different from zero can exist only for a codimension one set of coefficients c_{ijk}, and are generically absent, proving our claim.

Which of the three branches is turned on depends on the signs of the Fayet-Iliopoulos parameters ϑ_i:

- $x_j = 0$ corresponds to $\vartheta_3 < 0, \vartheta_2 > 0$, with ϑ_1 of either sign
- $y_k = 0$ corresponds to $\vartheta_1 > 0$, $\vartheta_2 < 0$, with ϑ_3 of either sign.
- $z_i = 0$ corresponds to $\vartheta_3 > 0$, $\vartheta_1 < 0$, with ϑ_2 of either sign.

Note that on any branch, eqs. (5.36-5.38) show that the nonzero $|x_j|^2, |y_k|^2, |z_i|^2$ are bounded by $|\vartheta_i|$ and hence can be made small, justifying our use of the cubic superpotential.

When (5.40) is satisfied, there is an even simpler argument for the absence of branches with all x, y, z nonzero: in this case $\partial_y W = 0$ imposes more equations than there are unknowns x_j, z_i, so this equation will generically not have any solutions apart from the trivial ones $x = 0$ or $z = 0$. Assuming $\vartheta_3 > 0$ (the case $\vartheta_3 < 0$ can be dealt with analogously) then implies using (5.38) that $x \neq 0$, so we must put $z = 0$. From (5.38) it then follows that there can only be solutions for $\vartheta_1 < 0$, decaying at marginal stability $\vartheta_1 = 0$ by splitting off Γ_1, in accordance with what we found in the spacetime picture in section 3.8.

Since $z = 0$, the D- and F-constraints reduce to

$$\mathcal{M} = \{(x,y) \in \mathbb{CP}^{a-1} \times \mathbb{CP}^{b-1} \,|\, \sum_{j,k} c_{ijk} x_j y_k = 0, \quad i = 1, \ldots, c\}, \tag{5.44}$$

with x and y now interpreted as homogeneous coordinates. Now for any fixed $x \in \mathbb{CP}^{a-1}$, the above equations cut out a $\mathbb{CP}^{b-c-1}$ in $\mathbb{CP}^{b-1}$. It is clear that this is true for *generic* x, but when (5.40) is satisfied (and c_{ijk} is generic, as we assume throughout), it will in fact be true for *any* x. This is easily seen for example by taking the c_{ijk} such that $M_{ik}(x) \equiv \sum_j c_{ijk} x_j \equiv x_{k-i+1}$ (with $x_j \equiv 0$ if j is outside the range $1, \ldots, a$), and noting that $M(x)$ manifestly has always maximal rank on $\mathbb{CP}^{a-1}$.

Therefore in this regime $\mathcal{M}$ is a $\mathbb{CP}^{b-c-1}$ fibration over $\mathbb{CP}^{a-1}$, without any fibers degenerating. Therefore

$$\chi(\mathcal{M}) = (b-c)\, a, \tag{5.45}$$

which brings (5.43) in exact agreement with (5.41).

Note that this is again an explicit realization of the picture outlined in section 4.1, and of the assumptions made there. In particular we see explicitly that the F-constraints effectively put the net intersection product between the two custers $\langle \Gamma_2 + \Gamma_3, \Gamma_1 \rangle = b - c$ equal to the total number of nonzero bifundamental scalars between theses two constituents, and that the fibration is regular.

It would be very interesting — even within the context of the present example — to do a more systematic study of the various flow trees that can arise and in which regions they do arise. It would also be interesting to compare more systematically with the quiver picture. This picture will be accurate in an open region around the point where all three charges Z_i are aligned, but it is easy to see that it must fail at some distance of order one from this point, because the lines $\vartheta_i = 0$ and $\theta_i = 0$ will not coincide. However, these matters lie outside the scope of this short note, so we will leave them for future work.

5.3 Entropy of the three node quiver in the scaling regime

The analysis of the three node closed loop quiver changes significantly when all three triangle inequalities are satisfied. For one thing, fiber jumps now become possible, with the potential of drastically increasing the complexity and Euler characteristic of $\mathcal{M}$. This is expected physically: as we saw in section 3.8, in this case there is no obstruction to letting the centers approach each other arbitrarily closely, asymptotically forming a black hole. Such solutions can no longer be forced to split. Thus they are not described by a flow tree but rather by a single flow, so our factorization arguments no longer apply, and the emergence of a horizon in the asymptotic limit suggests instead an exponential black hole type ground state degeneracy.

Note that our realization of the quiver described above and exemplified in figure 14 is never in this regime, since it always violates the triangle inequalities by $b > a + c$. It is not extremely easy to find realizations of scaling solutions in terms of rigid constituents. However, we expect that a simple realization with rigid nodes satisfying the triangle inequalities and corresponding to a scaling solution can be obtained by adding a rigid D2-brane to the charge Γ_3 in eq.(3.57). (We say "expect" because we have not verified that the discriminant $\mathcal{D}(H(\vec{x}))$ is everywhere positive.) We assume there are such realizations and proceed.

To compute the Euler characteristic in this case requires more sophisticated machinery. According to the general formulae of [94] the Euler characteristic is given in the general case by

$$\chi(\mathcal{M}) = \frac{\partial_{J_1}^{a-1}}{(a-1)!}\frac{\partial_{J_2}^{b-1}}{(b-1)!}\frac{(1+J_1)^a(1+J_2)^b}{(1+J_1+J_2)^c}(J_1+J_2)^c|_{J_1=J_2=0} \tag{5.46}$$

$$= \oint dJ_1 \oint dJ_2 J_1^{-a} J_2^{-b} \frac{(1+J_1)^a(1+J_2)^b}{(1+J_1+J_2)^c}(J_1+J_2)^c. \tag{5.47}$$

The contour integrals are on small contours $J_i = \epsilon_i e^{i\theta_i}$ and the relative sizes of ϵ_i do not matter.

The evaluation of these integrals is nontrivial and given in appendix E. We find the following elegant exact expression:

$$\chi(\mathcal{M}) = ab - \int_0^\infty ds\, e^{-s}\, L_{a-1}^1(s)\, L_{b-1}^1(s)\, L_{c-1}^1(s), \tag{5.48}$$

where the L_*^1 are Laguerre polynomials. Amusingly, these kinds of integrals arise in atomic physics, since the Laguerre polynomials are the radial eigenfunctions of an electron in a Coulomb potential. We don't know if this is a coincidence or has a deeper physical explanation. Recall that the small asymmetry between (a, b) and c arises from the fact that we are considering the case $\theta_1 < 0$, $\theta_3 > 0$, putting us on the branch $z = 0$. The other cases give rise to expressions with obvious modifications.

Equation (5.48) reproduces (5.45) when $b+1 \geq a+c$, so in particular also when (5.40) is satisfied, as expected. More interestingly, when all triangle inequalities are satisfied, we find that the degeneracies start increasing exponentially. This is in beautiful agreement with the fact that in this regime, the state no longer splits and a black hole can be formed, as we saw in section 3.8. More precisely, we find the remarkably simple and suggestive result

$$\chi(\mathcal{M}) \sim (abc)^{-1/3}\, 2^{a+b+c} \tag{5.49}$$

in the regime in which (a, b, c) are not too different from each other. Note that this amounts to a macroscopic entropy, since the intersection products (a, b, c) scale as Λ^2 when scaling up uniformly all charges by Λ, just like a large black hole.

The formula (5.49) suggests an interpretation in terms of fermionic degrees of freedom stretched between the centers, one per unit of intersection product, at least at large a, b, c. It is an interesting open problem to explain this result. Clearly, one would have a hard time getting such exponential degeneracies from a simple three particle quantum mechanics, unless new degrees of freedom appear in the scaling regime.

6. Counting BPS degeneracies

In this section we specialize the general tools developed so far to our actual problem of counting D4-D2-D0 state degeneracies. The idea is roughly as follows. We use the fareytail expansion to reduce the counting problem to computing polar D4-D2-D0 indices. We show, using our index factorization formulae, that at least for the "extreme" polar states, these indices factorize into D6 and anti-D6 indices. This leads to an approximate factorization of the leading term of the fareytail series into a D6 and an anti-D6 partition function. We argue that the latter can be identified with the topological string partition function. Putting the pieces together, we obtain an expression of the form $\mathcal{Z}_{\rm BH} \sim \mathcal{Z}_{\rm top}\overline{\mathcal{Z}_{\rm top}}$.

In section 6.1, we explore the relation of D6-D4-D2-D0 indices and DT invariants. This is nontrivial because of the background dependence of the former. We also show how quantizing particle halos leads to MacMahon and Gopakumar-Vafa-type generating functions. In section 6.2, we compute indices of D6-anti-D6 bound states using index factorization, and in section 6.3 we show that this leads to a suitably factorized generating function. In section 6.4 we show to what extent this can be used to compute polar D4-D2-D0 indices, and based on this we give a derivation of a refined version of the OSV conjecture (6.113).

6.1 D6-D4-D2-D0 degeneracies

We now turn more specifically to degeneracies of bound states of a single D6 with lower dimensional branes. Donaldson-Thomas invariants, which "count" ideal sheaves, count in some sense BPS bound states of a single D6 with D2- and D0-branes. However, there is an immediate problem with this interpretation: the actual BPS indices of such bound states depends strongly on the background moduli, due to jumping phenomena at marginal stability walls, while DT invariants are insensitive to the background. We saw examples of the background dependence of such BPS states in sections 3.6 and 5. Examples of supergravity realizations of such states with a limited domain of stability are halos of D0-branes around some D6-D4-D2-D0 core. More general such halo-configurations exist, for example replacing the D0-particles by D2-D0 particles, as shown in fig. 7*a*. The latter have an even richer structure of marginal stability walls, and they extend all the way to the infinite radius limit, so unlike for D4-D2-D0 systems, one cannot avoid these issues by restricting to the large radius limit. Moreover, as is manifest e.g. in equation (5.4), when these D6-D4-D2-D0 bound states are used as building blocks for more complicated configurations or flow trees, the relevant moduli are determined by the split points of the attractor flow trees, so we cannot just pick some values we happen to like.

6.1.1 D6 + D0-halos

Let us begin by considering the degree zero part $\mathcal{Z}^0_{DT}$ of the DT partition function, introduced in (1.21). This supposedly counts D6-D0 bound states, but we know that at $B = 0$ for example, there are no such bound states. On the other hand, for B sufficiently large, these bound states do exist, and then $\mathcal{Z}^0_{DT}$ correctly counts them.

A simple way to derive this is through the D0-halo picture in supergravity. The following gives a sketch of how this is done, based on the detailed study of analogous systems in [65], to which we refer for more details.

Since D0-branes can form bound states with each other of arbitrary charge, the particles in the halo can have arbitrary D0-charge. To have a BPS configuration, all charges have to be of the same sign though, determined by the sign of the B-field, as in (3.55). Let us assume this is positive.

As reviewed under (3.24), the contribution to the angular momentum of a particle of D0-charge n moving in the magnetic field of a D6, arising from its position degrees of freedom and the intrinsic monopole-electron type angular momentum stored in the electromagnetic field, equals $j = \frac{1}{2}\langle D6, nD0\rangle - \frac{1}{2} = (n-1)/2$, hence the contribution to the degeneracy of BPS ground states from quantizing these degrees of freedom equals $2j + 1 = n$.

In addition, the particle has a number of "internal" BPS ground states, obtained by quantizing its position moduli (super)space inside the Calabi-Yau threefold X. These are simply given in the usual way by the cohomology of X, and their spin is determined by the Lefshetz SU(2) action on $H^*(X)$; in particular a p-form has spin $j_3 = (p-3)/2$, which is half-integral when p is even. Since as we saw, the $\mathbb{R}^3$ position hypermultiplet is forced by the radial magnetic field to be in a spin 1/2 state, the even cohomology will thus correspond to bosonic particles, and the odd cohomology to fermionic particles.[40]

[40] Here "bosonic" and "fermionic" refers to the nature of the individual particles (the electrons). Whether the bound state with the D6 as a whole (the atom) will be fermionic or bosonic also depends on the spin $(n-1)/2$ coming from the quantization of the monopole-electron system as discussed above.

To summarize, one particle ground states are labeled by their D0-charge $n \geq 1$, an integer $m \in \{0, \dots, n-1\}$ labeling the lowest Landau levels, and an element ω of $H^*(X)$. Since the particles in the halo are mutually BPS, classically they do not exert static forces on each other, and hence in the $\hbar \to 0$ limit the multiparticle ground states are simply labeled by occupation numbers of single particle states: $|\{k_{n,m,\omega}\}_{n,m,\omega}\rangle$, where ω runs over a basis for the cohomology. Since we are considering a Fock space of D0 particles in a fixed D6 background we form a Fock space of bosonic one particle states corresponding to even degree cohomology classes and of fermionic one particle states corresponding to odd degree cohomology classes. Thus we have bosonic occupation numbers $k_{n,m,\omega_e} \in \mathbb{N}$, where ω_e runs over a basis for the even-degree cohomology and fermionic occupation numbers $k_{n,m,\omega_o} \in \{0,1\}$, where ω_o runs over a basis for odd-degree cohomology. While the fermi/bose nature of the individual particles is governed by the degree of the cohomology class, the total spin J_3' (as usual with the contribution of the center of mass degrees of freedom factored out) is $J_3' = \sum k_{n,m,\omega}(m - \frac{n-1}{2} + \frac{\deg(\omega)-3}{2})$. Putting all this together, and letting b_e, b_o denote the dimension of the even, respectively odd cohomology, the generating function for the index d_N of D6-D0 BPS bound states of total D0-charge N (defined following (1.6)-(1.7)) is

$$
\begin{aligned}
\sum_N d_N\, u^N &= \mathrm{Tr}\,(-1)^{2J_3'}\, u^N \\
&= \sum_{k_{n,m,\omega_e}} \sum_{k_{n,m,\omega_o}} (-1)^{\sum k_{n,m,\omega_o} + \sum n(k_{n,m,\omega_e}+k_{n,m,\omega_o})}\, u^{\sum n(k_{n,m,\omega_e}+k_{n,m,\omega_o})} \\
&= \prod_{n=1}^{\infty} \left(\sum_{k=0}^{\infty} (-u)^{nk} \right)^{nb_e} \left(\sum_{k=0}^{1} (-1)^k (-u)^{nk} \right)^{nb_o} \\
&= \prod_{n=1}^{\infty} (1-(-u)^n)^{n(-b_e+b_o)} = \prod_{n=1}^{\infty} (1-(-u)^n)^{-n\chi(X)} = M(-u)^{\chi(X)} \qquad (6.1)
\end{aligned}
$$

where M is the MacMahon function. This exactly reproduces the expression (1.21) for $\mathcal{Z}^0_{DT}$, including all signs.

If the value of the B field is such that the BPS condition requires negative $D0$ charge, the generating function is obtained from the one above by substituting $u \to u^{-1}$.

Incidentally, from the weak coupling expansion (1.16) of the MacMahon function — in particular from the $\exp[\frac{\chi(X)\zeta(3)}{2g^2}]$ singularity — we can extract the $n \to \infty$ asymptotics

$$
d_n = N_{DT}(0,n) \sim \begin{cases} \exp[\frac{3}{2}(\chi\zeta(3)n^2)^{1/3} + (\frac{1}{2} - \frac{\chi}{72})\log n] & \text{if} \quad \chi(X) > 0 \\ \mathrm{Re}\Big[e^{i\phi}\exp[e^{i\pi/3}\frac{3}{2}(|\chi|\zeta(3)n^2)^{1/3} + (\frac{1}{2} - \frac{\chi}{72})\log n]\Big] & \text{if} \quad \chi(X) < 0 \end{cases} \qquad (6.2)
$$

where ϕ is a real constant and we dropped the 1-loop prefactor. Hence the large n entropy of the D0-halo goes roughly like $S \sim n^{2/3}$, but with large oscillatory fluctuations when $\chi(X) < 0$.

As we saw in section 3.7, D0-halos appear as part of multicentered D4D2D0 states, and thus the MacMahon function naturally appears in generating functions determining black hole entropies. In this way the above derivation resolves an old puzzle. It seemed mysterious how microstate counting could account for the strange term $\chi(X)\zeta(3)/g^2$ in the expansion of the topological free energy $F_{\rm top}$ and therefore in the black hole entropy formula. Now we see where it comes from: the MacMahon function arises from counting D0-halo states, and this in turn gives rise to the term $\chi(X)\zeta(3)/g^2$ from the small g asymptotic expansion (1.16).

6.1.2 D6 + D2-D0-halos and relation between BPS indices and DT invariants

Before we get to counting bound states of D6-branes with D2-D0 halos around them, let us briefly review the counting of D2-D0 BPS states and the definitions of BPS and DT invariants.

At zero string coupling, single D2-D0 particle states of $(D2, D0)$ charge (Q, n) are given by cohomology classes of D2 moduli space. The $(D2, D0)$ moduli space is a torus fibration over the deformation moduli space of a holomorphic curve in homology class Q in X. The curve represents the supersymmetric cycle on which $D2$ is wrapped while the torus fiber accounts for Wilson line moduli. The cohomology can be decomposed according to representations of the $SU(2)_L \times SU(2)_R$ Lefshetz action on moduli space [32, 33, 95, 111] where roughly $SU(2)_R$ acts on the base cohomology while $SU(2)_L$ acts on the fiber cohomology. After uplifting to M-theory, the $SU(2)$'s can be identified with the factors of the 5d little group $SO(4) = SU(2)_L \times SU(2)_R$ [91].

Let $N_Q^{m_L,m_R}$ be the dimension of the cohomology group of the moduli space D2 of branes of charge Q and $(J_L^3, J_R^3) = (m_L, m_R)$. One can construct a well-behaved index from this by tracing over the $SU(2)_R$ factor:

$$N_Q^{m_L} := \sum_{m_R} (-1)^{2m_R} N_Q^{m_L,m_R}. \tag{6.3}$$

The usual Witten index of all BPS states, which up to a sign is the Euler characteristic of the moduli space, is obtained by tracing this index in turn over the $SU(2)_L$ factor:

$$N_Q := \sum_{m_L} (-1)^{2m_L} N_Q^{m_L}. \tag{6.4}$$

These indices are related to the BPS invariants n_Q^r by [95]

$$N_Q^{m_L} = \sum_{r \geq |2m_L|} \binom{2r}{r + 2m_L} n_Q^r\,, \qquad N_Q = n_Q^0. \tag{6.5}$$

Note the interesting cancelation leading to the last expression, due to the binomial formula $\sum_{m=0}^n (-1)^m \binom{n}{m} = (1-1)^n = 0$. More fundamentally this follows from the fact that a D2-brane of genus greater than 0 comes with a moduli space containing a torus factor from the Wilson lines, which has zero total Euler characteristic when excluding degeneration limits. Only a complete degeneration to $g = 0$ components eliminates the torus fiber and gives rise to a nonzero Euler characteristic.

With these invariants and using the GW-GV-DT correspondence outlined in section 1.3, one can build up a generating function for DT invariants, as follows

$$\mathcal{Z}_{DT}(u, v) := \sum_{\beta,n} N_{DT}(\beta, n)\, u^n\, v^\beta = \prod_{Q>0, m_L, k>0} \left(1 - (-u)^{k+2m_L} v^Q\right)^{k(-1)^{2m_L} N_Q^{m_L}}. \tag{6.6}$$

After some manipulations starting from (6.5) and involving binomial identities and changing product variables, this can also be written as [34]

$$\mathcal{Z}_{DT}(u,v) = \mathcal{Z}^0_{DT}(u,v)\,\mathcal{Z}'_{DT}(u,v) \tag{6.7}$$

$$\mathcal{Z}^0_{DT}(u,v) = \prod_{k>0}(1-(-u)^k)^{-k\chi(X)} \tag{6.8}$$

$$\mathcal{Z}'_{DT}(u,v) = \mathcal{Z}'^{,r=0}_{DT}(u,v)\,\mathcal{Z}'^{,r>0}_{DT}(u,v) \tag{6.9}$$

$$\mathcal{Z}'^{,r=0}_{DT}(u,v) = \prod_{Q>0,k>0}(1-(-u)^k v^Q)^{k n^0_Q} \tag{6.10}$$

$$\mathcal{Z}'^{,r>0}_{DT}(u,v) = \prod_{Q>0,r>0}\prod_{\ell=0}^{2r-2}\Big(1-(-u)^{r-\ell-1}v^Q\Big)^{(-1)^{r+\ell}\binom{2r-2}{\ell}\,n^r_Q}. \tag{6.11}$$

The DT invariants count in some sense D6-D2-D0 bound states, as we will make precise below, but we can also use them to count more general D6-D4-D2-D0 bound states (with $p^0=1$), by parametrizing the charge Γ as in (4.17):

$$\Gamma = e^S\,\Gamma(\beta,n) := e^S(1-\beta+n\,\omega)(1+\tfrac{c_2}{24}) = e^S(1-\beta+\tfrac{1}{24}c_2+n\,\omega). \tag{6.12}$$

Multiplying by e^S amounts to tensoring by a line bundle with field strength $F=S$, or gauge equivalently, to shifting $B\to B-S$. The gauge invariant statement is that this transformation turns on a nonzero $\mathcal{F}:=F-B$. It is well known that tensoring by a line bundle this does not affect μ-stability. Essentially this argument was used in [44, 29] to conclude that $\Omega(\Gamma)|_t$ does not depend on S. However, μ-stability does not precisely coincide with physical stability, not even at infinite radius,[41] and indeed as we saw (and are going to elaborate on in what follows) the large radius BPS spectrum of D6-D2-D0 bound states is in fact *not* invariant under arbitrary shifts of $\mathcal{F}$, so the physical $\Omega(\Gamma)|_t$ will actually depend on S. Thus, the question arises then in which regime, if any, the DT invariants do count physical D6-D2-D0 BPS states.

Another issue, already mentioned in section 4.3, is that not only ideal sheaves $\mathcal{I}$ are suitable to model D6-D2-D0 bound states with $p^0=1$, but their duals $\mathcal{I}^*$ are as well. They differ for example in that ideal sheaves have D0 charge bounded from below at any fixed D2 charge, whereas their duals have D0 charge bounded above. This leads to the puzzle which of the two we should consider.

Both of these conundrums are resolved if the DT invariants correspond to BPS invariants only for suitable limits of the B field. In particular, we will consider limits in which B is taken proportional to J and taken to plus or minus infinity. The dichotomy between ideal sheaves and their (derived) duals then depends on the sign of the B-field.

To be specific, let us assume P is some arbitrary auxiliary class inside the Kähler cone, and

$$B+iJ=(x+iy)P, \qquad F=S=s\,P, \qquad \mathcal{F}:=F-B=(s-x)P=:f\,P. \tag{6.13}$$

Then we claim that any ideal sheaf with fixed (β,n) specified as in (6.12) will become stable for sufficiently large negative f, and their degeneracies $\Omega(\Gamma)|_{(x+iy)P}$ counted by the

[41] Thus disproving a conjecture made in [59].

DT invariants $N_{DT}(\beta, n)$, while for sufficiently large positive f, the duals of ideal sheaves are stabilized, and their degeneracies counted by $N_{DT}(\beta, -n)$.

At the end of this subsection, we will outline an argument for the correctness of this claim by uplifting to M-theory, refining the analysis of [29]. Before we get to this, we will make our claim more precise and elaborate on its consequences in the IIA picture.

First note that from the discussion of D6-D0 bound states above, it follows immediately that this proposal is correct for $\beta = 0$. When $\beta \neq 0$, there are in general various possible configurations with the same charge, consisting of a core which could for example be a simple D6-D4-D2-D0 black hole, surrounded by halos of D2-D0 particles at radii fixed by the D2-D0 charges and the background Kähler moduli. The typical state will thus look like an onion with many different layers of D2-D0 halos, as illustrated in fig. 15*a*.

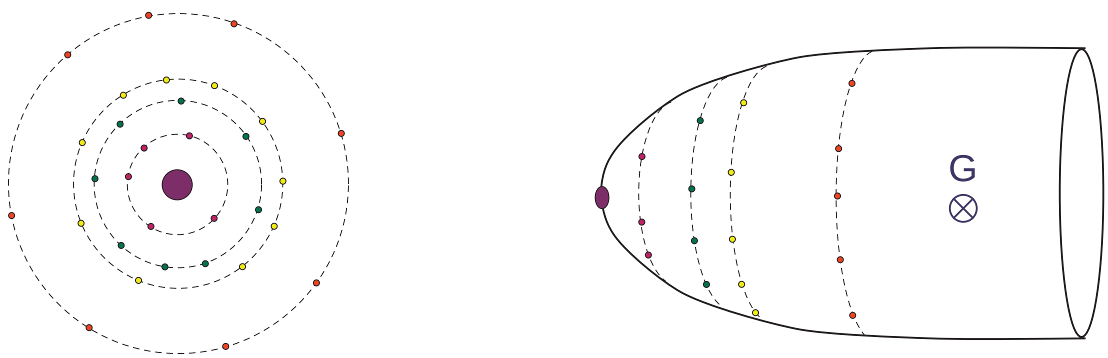


Figure 15: Left: Sketch of a typical D6-D4-D2-D0 bound state in 4d, consisting of layers of D2-D0 halos around a D6-D4-D2-D0 core (e.g. a black hole). The larger $\mathcal{F} = F - B$ is, the more layers can be added. Conversely, by decreasing $\mathcal{F}$, layers get peeled off one by one, moving out to $r = \infty$. **Right:** Uplift to 5d: M2 branes in Taub-NUT + flux (see below).

Halo configurations have walls of marginal stability and only exist for a certain range of values of $\mathcal{F}$. Applying the stability condition (3.23) to a two-particle system of total charge $\Gamma = e^S \Gamma(\beta, n)$, consisting of a core of charge $\Gamma_c = e^S \Gamma(\beta_c, n_c)$ around which a D2-D0 halo of charge $\Gamma_h = e^S(-\beta_h + n_h \omega)$ is orbiting, with S, B and J as in (6.13), we find

$$- n_h \left(2(P \cdot \beta_h) \left(f(f^2 + y^2) - \frac{3\,n}{P^3} \right) + n_h \left(y^2 - 3f^2 + \frac{6\,P \cdot \beta}{P^3} \right) \right) > 0. \tag{6.14}$$

Asymptotically for $y \to \infty$ this becomes

$$- n_h \left(2(P \cdot \beta_h) f + n_h \right) > 0, \tag{6.15}$$

so there is a marginal stability line running all the way to infinity, at $x = s + n_h / 2(P \cdot \beta_h)$. Actually for this to be a true marginal stability line where the phases of the halo and core central charges align, we also need $\beta_h \cdot P > 0$, as can be seen by examining the asymptotic behavior of the central charges for $J \to \infty$.

Note in particular that (6.15) implies that at $f = 0$ (and $y \to \infty$), there are *never* such BPS states, while for $f \to -\infty$, all $n_h > 0$ states become stable, while all $n_h < 0$

states become stable in the opposite regime $f \to +\infty$. In fact the latter is also true at finite values of y, as is easily seen from (6.14).

When we add more D2-D0 particles, we should in principle use the more general multicentered stability conditions (3.21). Since the mutual intersection products between the D2-D0 particles are zero, this effectively boils down again to the 2-centered stability condition we used to obtain (6.14) for each individual particle in the halo, with $e^S(-\beta_h + n_h\omega)$ the halo particle charge considered and $e^S\Gamma(\beta, n)$ the total charge of the system.

Note though that for $y \to \infty$ the (β, n) dependence in (6.15) drops out so the stability conditions are given by a set of simple, independent constraints. Similarly, independent of which halo particles are present, it will always remain true that for $x \to +\infty$, $n_h > 0$ halos are stabilized, while for $x \to -\infty$, $n_h < 0$ halos are stabilized. Hence we see that the large radius spectrum of D6-D2-D0 bound states has total D0-brane charge n unbounded above at $x \to +\infty$, while n is unbounded below at $x \to -\infty$. The former is characteristic for ideal sheaves, while the latter is characteristic for their duals. (Recall eq. 4.24.) This supports our claim above.

One could ask how exactly the index of BPS states $\Omega(\Gamma)|_{B+iJ}$ changes when we change $\mathcal{F}$. This is most easily described by considering the generating function

$$\mathcal{Z}_{\rm D6-D2-D0}(u, v; B + iJ) := \sum_{\beta,n} \Omega(\Gamma(\beta, n))|_{B+iJ}\ u^n\, v^\beta \tag{6.16}$$

where $v^\beta \equiv \prod_A (v_A)^{\beta_A}$ and we take $\Gamma(\beta, n)$ as in (6.12) (so here $\mathcal{F} = -B$). We will in particular be interested in the case $B + iJ = (x + iy)P$, with P at this point an arbitrary auxiliary class inside the Kähler cone.

The generating function $\mathcal{Z}_{\rm D6-D2-D0}(u, v; (x + iy)P)$ will jump whenever $x + iy$ is changed such that (6.14) goes from not being satisfied to being satisfied (or vice versa) for some (β_h, n_h). This adds states consisting of an arbitrary[42] number of D2-D0 particles (consistent with the exclusion principle of course if the particles are fermions), leading, following a reasoning similar to the derivation of the D0-halo degeneracy to a jump

$$\mathcal{Z}_{D6-D2-D0} \to (1 - (-u)^{n_h} v^{\beta_h})^{|n_h| N_{\beta_h}}\, \mathcal{Z}_{D6-D2-D0}, \tag{6.17}$$

with $N_{\beta_h} = n^0_{\beta_h}$ as in (6.4) and (6.5). Analogous to the D6-D0 system, the factor $|n_h|$ comes from the Landau degeneracy of the D2-D0 particle in the D6 background, since the intersection product equals $|n_h|$. Using this, and recalling that at $x \to +\infty$ all $n_h > 0$ halos are stabilized, while at $x = 0$, $y = \infty$ none are, we can write

$$\begin{aligned}\lim_{x\to+\infty} \lim_{y\to+\infty} \frac{\mathcal{Z}_{D6-D2-D0}(u, v; (x + iy)P)}{\mathcal{Z}_{D6-D2-D0}(u, v; iyP)} &= \prod_{\beta_h>0, n_h>0} (1 - (-u)^{n_h} v^{\beta_h})^{n_h n^0_{\beta_h}} &(6.18)\\ &= \mathcal{Z}'^{,r=0}_{DT}(u, v). &(6.19)\end{aligned}$$

When the limits are interchanged, there is an additional factor for $\beta_h = 0$ given by (6.1), corresponding to D0-halos.

[42]Note that (6.14) is invariant under $(\beta, n) \to (\beta, n) + k(\beta_h, n_h)$, as it should, so the number of particles of charge (β_h, n_h) we add does not matter for stability.

Comparing to (6.7)-(6.11), we see that our proposed identification of

$$\lim_{x\to+\infty}\lim_{y\to+\infty}\mathcal{Z}_{D6-D2-D0}(u,v;(x+iy)P)$$

with $\mathcal{Z}'_{DT}(u,v)$ is valid provided

$$\lim_{y\to\infty}\mathcal{Z}_{D6-D2-D0}(u,v;iyP)=\mathcal{Z}'^{,r>0}_{DT}(u,v). \tag{6.20}$$

Note that this is manifestly invariant under $u\to u^{-1}$ (i.e. inversion of D0-charge) and has a finite range of D0-charge for fixed β_h. Similarly, if we require our proposed identification in the opposite regime, namely $\lim_{x\to-\infty}\lim_{y\to+\infty}\mathcal{Z}_{D6-D2-D0}(u,v;(x+iy)P)=\mathcal{Z}'_{DT}(u^{-1},v)$, to hold, we find again (6.20).

In fact, for our analysis below, we will need to refine these statements. There are four distinct ways of taking a limit to infinity, due to the fact that D6D0-type lines of marginal stability go all the way to infinity, being asymptotically of the form $z:=x+iy=\lambda e^{2\pi i/3}$ and $z=\lambda e^{\pi i/3}$ with $\lambda\to+\infty$. Correspondingly, we distinguish the limits $z\to\mathcal{L}^+$, along a line infinitesimally above $z=\lambda e^{2\pi i/3}$, and $z\to\mathcal{L}^-$, along a line infinitesimally below $z=\lambda e^{2\pi i/3}$. Similarly, $z\to\mathcal{R}^\pm$ is the limit going infinitesimally above (below) the line $z=\lambda e^{\pi i/3}$. Now we have

$$\lim_{z\to\mathcal{L}^-}\mathcal{Z}_{D6-D2-D0}(u,v;zP)=\mathcal{Z}_{DT}(u^{-1},v) \tag{6.21}$$

$$\lim_{z\to\mathcal{L}^+}\mathcal{Z}_{D6-D2-D0}(u,v;zP)=\mathcal{Z}'_{DT}(u^{-1},v) \tag{6.22}$$

$$\lim_{z\to\mathcal{R}^-}\mathcal{Z}_{D6-D2-D0}(u,v;zP)=\mathcal{Z}_{DT}(u,v) \tag{6.23}$$

$$\lim_{z\to\mathcal{R}^+}\mathcal{Z}_{D6-D2-D0}(u,v;zP)=\mathcal{Z}'_{DT}(u,v) \tag{6.24}$$

Finally, we sketch how to establish the proposed identification by uplifting to M-theory, refining the analysis of [29]. The lift of D6-D4-D2-D0 bound states to M-theory is given by M2 branes in Taub-NUT (deformed by the flux) times the Calabi-Yau, with the D0-charge corresponding to the $U(1)_L$ isometry along the Taub-NUT circle. This is sketched in fig. 15*b*. Turning on wordvolume flux $F=S=sP>0$ on the D6 corresponds to turning on an M-theory 4-form flux $G=sP\wedge\omega_{TN}$, where ω_{TN} is the harmonic self-dual 2-form on Taub-NUT. In the absence of this flux, the M2 branes must sit at the center of Taub-NUT to be BPS. This corrseponds to the core states. Now when we turn on the magnetic flux, the M2 branes get access to a number of lowest Landau level states, carrying increasing $U(1)_L$ charge. These will in general be localized at a nonzero distance from the center of Taub-NUT, and thus correspond to the halo states in four dimensions. However, for a finite total integrated flux, there is a bound on the number of lowest Landau levels that fit in the Taub-NUT space — beyond this cutoff, the equilibrium location of the M2 branes runs off to radial infinity. The number of lowest Landau levels that do fit in the Taub-NUT space is proportional to the flux, and this leads to the dependence of the BPS spectrum on the flux derived above in the IIA picture. The limit $|s|\to\infty$ corresponds to infinite total integrated flux, which removes the bound on allowed lowest Landau levels. If at the same time we let the Taub-NUT radius go to infinity, we end up with a constant, arbitrarily small flux density $G \sim i(dz_1 \wedge d\bar{z}_1 + dz_2 \wedge d\bar{z}_2) \wedge P$ on flat $\mathbb{C}^2 \times X$, which is indeed the background implicitly used in [29] to show that M2 BPS states are counted by $\mathcal{Z}_{top} = \mathcal{Z}_{GV} = \mathcal{Z}_{DT}$. Therefore, in the $|s| \to \infty$ regime — and only in this regime — the derivation of [29] goes through, proving our claim.

6.1.3 Core states

In the previous section we were led to interpret the factor $\mathcal{Z}'^{,r=0}_{DT}(u,v)$ as counting halo states. It is natural to wonder about the physical interpretation of the remaining, non-halo states counted by $\mathcal{Z}'^{,r>0}_{DT}(u,v)$ in (6.20). We will refer to these states as *core states*. Core states are characterized by the absence of marginal stability walls extending to infinite radius.

One immediate consequence is that at sufficiently large background J, these states are stable for all values of the B-field. Single centered black holes are of course the simplest example, but multicentered configurations are also possible as we will show below. In this case the centers can be squeezed arbitrarily close together (at least in coordinate distance) by taking J sufficiently large. These states can subsequently be "dressed" with the D2-D0 halos described in the previous subsection, which even at infinite J can be given arbitrarily large radius by tuning the B-field close to the wall of marginal stability. This justifies the names core and halo states.

Another consequence is that core states, unlike halo states, have degeneracies at $J \to \infty$ symmetric under inversion of D0-charge. This is a result of combining the $\Gamma \to \Gamma^*$, $B \to -B$ symmetry described in section 3.3 with the absence of walls of marginal stability for $J \to \infty$. This is in agreement with the fact that $\mathcal{Z}'^{,r>0}_{\mathrm{DT}}$ is invariant under $u \to u^{-1}$, while $\mathcal{Z}'^{,r=0}_{\mathrm{DT}}$ is not.

Note that multicentered bound states of charges which all have nonzero magnetic (D4 or D6) charges are always core states. In other words halos with lines of marginal stability extending to infinite radius can never contain magnetic charge. For halo particles with nonzero D4 but zero D6 charge this follows from the fact that since the total charge Γ must have D6-charge 1 (by assumption), at $J = \infty$ we have $Z(\Gamma) \sim -iJ^3$ imaginary while $Z(D4) \sim P \cdot J^2$ is real, so there cannot possibly be a wall of marginal stability for splitting off the D4 extending all the way to $J = \infty$. For halo particles with negative D6 charge centers the reasoning is similar: now the central charges both are imaginary at infinity, but with opposite phases. For halo particles with positive D6 charge the complement has necessarily negative or zero D6 charge, so the previous reasoning can be applied to the central charge of the complement. Only when the complement has nothing but D2-D0 charge (with $P \cdot Q_{D2} < 0$) is there a wall of marginal stability which extends to infinity, but in this case the original center of course corresponds again to a core state, the complement being a D2-D0 particle or halo orbiting around it.

An example of a nontrivial core state was in fact already discussed above in section 5.2.2. We will now examine another class of core states, which we will use to construct "swing states" in section 6.3.2. These will play an important role in delimiting the region of validity of the OSV conjecture. For this reason, we will give a detailed stability analysis of this class.

We consider bound states of a pure fluxed D6 with a D4-D2-D0 black hole, such that the total charge Γ has no D4 charge, that is we take (neglecting the $c_2/24$ correction)

$$\Gamma = \Gamma(\beta_1, n_1) = 1 - \beta_1 + n_1\,\omega \tag{6.25}$$

constructed as a bound state

$$\Gamma = \tilde{\Gamma} + \tilde{\Gamma}' \tag{6.26}$$

with

$$\tilde{\Gamma} = e^{-U-V}(U + q_0\,\omega) \qquad \& \qquad \tilde{\Gamma}' = e^{-U} \tag{6.27}$$

so

$$\beta_1 = UV + \frac{1}{2}U^2 \qquad \& \qquad n_1 = \frac{1}{2}U(U+V)^2 - \frac{1}{6}U^3 + q_0\,\omega. \tag{6.28}$$

To be more specific, we assume that U, V are positive divisors, which for simplicity we take to be proportional to P. This allows us to restrict the moduli to the complex plane $B + iJ = z\,P$, rendering the problem effectively one dimensional. It is furthermore convenient to define the notation $U = uP, V = vP, q_0 = \tilde{q}_0 P^3$. We will assume that $\tilde{\Gamma}$ is realized as a single centered black hole, which amounts to taking $\tilde{q}_0 < 0$. We will more specifically be interested in cases with small u and v of order 1, with $|\tilde{q}_0|$ sufficiently small so

$$\frac{1}{2}uv^2 + \tilde{q}_0 > 0. \tag{6.29}$$

An example is shown in fig. 16(a).

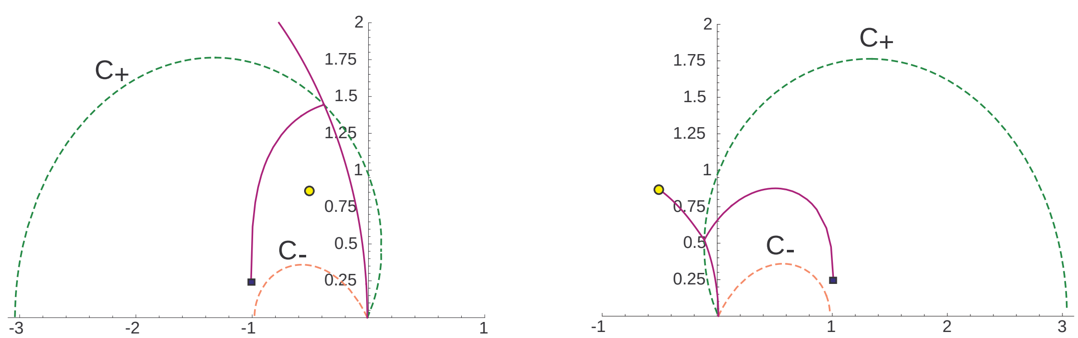


Figure 16: Left: Plot in the z-plane of the split flow described in the text with $u = 10^{-2}$, $v = 1$, $q_0 = -10^{-4}$, $z_\infty = 3e^{2i\pi/3}$. The green dotted line labeled C_+ is the ms line, the pink dotted line labeled C_- the anti-ms line. Note that at $z_\infty = e^{2i\pi/3}$ (yellow dot), the bound state does not exist. **Right:** Split flow for dual charges obtained by setting $u = -10^{-2}$, $v = -1$, $q_0 = 10^{-4}$, with $z_\infty = e^{2\pi i/3}$. We do not analyze such negative u, v cases in detail in the text because unlike the positive (u, v) cases, their stability for z_∞ near $e^{2i\pi/3}$ is guaranteed, making them less of an issue in the analysis in the following sections.

Let us analyze for what values of z_∞ this state exists. Since $\tilde{\Gamma}$ is a single centered black hole and $\tilde{\Gamma}'$ is just a pure $D6$ with flux, the two constituents are guaranteed to exist everywhere in moduli space. Therefore the stability analysis reduces to an analysis of the existence and location of a line of marginal stability for the split flow $\Gamma \to \tilde{\Gamma} + \tilde{\Gamma}'$.

The total discriminant is

$$8u^3(v+u/2)^3 - 9(uv^2/2 + u^2 v + u^3/3 + \tilde{q}_0)^2 \tag{6.30}$$

up to a positive coefficient of proportionality. Hence, when v is of order 1 and u is small, $\mathcal{D}(\Gamma)$ will be negative. This means there will be a zero of the central charge, which should be reached by the Γ attractor flow only after crossing a marginal stability line if we want the split flow to exist.

Now we consider the stability condition (3.23). Define $\hat{z} := z + u$ so that the central charges are:

$$Z(\tilde{\Gamma}; B+iJ) = -\frac{1}{2}u((\hat{z}+v)^2 + \frac{2\tilde{q}_0}{u})P^3 \qquad \& \qquad Z(\tilde{\Gamma}'; B+iJ) = \frac{\hat{z}^3}{6}P^3 \tag{6.31}$$

so the stability condition becomes

$$-(\frac{1}{2}uv^2 + \tilde{q}_0)\,\mathrm{Im}\Big[(\frac{1}{2}u(\hat{z}+v)^2 + \tilde{q}_0)\bar{\hat{z}}^3\Big] > 0 \tag{6.32}$$

The marginal stability curve

$$\mathrm{Im}\Big[(\frac{1}{2}u(\hat{z}+v)^2 + \tilde{q}_0)\bar{\hat{z}}^3\Big] = 0 \tag{6.33}$$

can be written as:

$$(x^2+y^2)^2 + 4v(x^2+y^2)x + (v^2 + \frac{2\tilde{q}_0}{u})(3x^2 - y^2) = 0$$

where $\hat{z} =: x + iy$. The solution is the x-axis together with two bounded components roughly of the shape of a cardiode with the tip at the origin $\hat{z} = 0$. Writing $\hat{z} = re^{i\theta}$ they are given by

$$r = -2v\cos(\theta) \pm \sqrt{v^2 + \frac{2\tilde{q}_0}{u}(1 - 4\cos^2(\theta))} \tag{6.34}$$

Call the plus branch $\mathcal{C}_+$ and the minus branch $\mathcal{C}_-$.

Under the condition (6.29) we find that (in the upper half plane) $\mathcal{C}_+$ is swept out $\frac{\pi}{3} \le \theta \le \pi$ and $\mathcal{C}_-$ is swept out for $\frac{2\pi}{3} \le \theta \le \pi$. Clearly $\mathcal{C}_+$ encloses $\mathcal{C}_-$ and they only intersect at the origin. The region at $r \to \infty$ is a region of stability.

Under our conditions $\mathcal{C}_+$ is indeed a line of positive marginal stability and $\mathcal{C}_-$ is a line of anti-marginal stability, and in fact the entire inside region of $\mathcal{C}_+$ is a region of instability.

(To prove the above statement, we note that the lines $\mathrm{Im}Z(\tilde{\Gamma})\overline{Z(\tilde{\Gamma}')}$ and $\mathrm{Re}Z(\tilde{\Gamma})\overline{Z(\tilde{\Gamma}')}$ can only intersect when the product of central charges is zero, that is, at $\hat{z} = 0$ or at $\hat{z} = -v \pm \sqrt{\frac{-2\tilde{q}_0}{u}}$. However the curve $\mathrm{Re}Z(\tilde{\Gamma})\overline{Z(\tilde{\Gamma}')} = 0$ intersects the x axis only at $x = y = 0$ and at $\hat{z} = -v \pm \sqrt{\frac{-2\tilde{q}_0}{u}}$. Now $\mathcal{C}_+$ intersects the x axis at $x_+ = -2v - \sqrt{v^2 - \frac{6\tilde{q}_0}{u}}$ and since

$$x_+ < -v - \sqrt{\frac{-2\tilde{q}_0}{u}} \tag{6.35}$$

it follows that the change of sign of $\mathrm{Re}Z(\tilde\Gamma)\overline{Z(\tilde\Gamma')}$ happens inside the region enclosed by $\mathcal{C}_+$. Finally, note that up to a positive coefficient $\mathrm{Re}Z(\tilde\Gamma)\overline{Z(\tilde\Gamma')}$ is given by

$$-\left[(x^2+y^2)^2x+2v(x^4-y^4)+(v^2+\frac{2\tilde q_0}{u})(x^3-3xy^2)\right] \tag{6.36}$$

and hence equals

$$-\frac{1}{2}ur^5\cos(\theta)+\mathcal{O}(r^4)$$

for large r, so clearly if $\theta>\pi/2$, $r\to\infty$ the quantity is positive.)

It follows that if z_∞ is outside the compact region enclosed by $\mathcal{C}_+$ then the split state does exist: The zero of $Z(\Gamma;zP)$ lies on the antimarginal stability curve $\mathcal{C}_-$ which is contained within $\mathcal{C}_+$. The attractor flow heads toward this zero, and splits on the line $\mathcal{C}_+$. On the other hand, if z_∞ is inside the curve $\mathcal{C}_+$ then the split state does not exist.

Note that the M-theory uplift of such a 2-centered configuration is a black ring orbiting the center of a Taub-NUT space with flux, obtained from wrapping an M5 around a divisor U and the Taub-NUT circle and giving it some momentum q_0 around the Taub-NUT circle [71, 76, 77]. The fact that the discriminant of the total charge is negative means that this charge cannot be realized as a BMPV black hole in Taub-NUT.

6.2 D6-anti-D6 degeneracies

6.2.1 Spectrum and flow trees

We now turn to our main goal, namely computing degeneracies of polar D4-D2-D0 BPS states represented as D6-anti-D6 bound states. The attractor flow trees corresponding to those can be as simple as fig. 2a or as complex as fig. 8, but in any case, the first split will be into a pair of charges with D6-charges r and $-r$ with $r>0$.[43] The case $r=1$ will turn out to be the most important one, so let us consider pairs of charges Γ_1 and Γ_2, parametrized as in (4.17)-(4.19), i.e.

$$\Gamma_1 = e^{S_1}\Gamma(\beta_1,n_1)=e^{S_1}(1-\beta_1'+n_1\,\omega), \qquad \beta_1':=\beta_1-\tfrac{c_2}{24} \tag{6.37}$$

$$\Gamma_2 = -e^{S_2}\Gamma(\beta_2,n_2)=-e^{S_2}(1-\beta_2'+n_2\,\omega), \qquad \beta_2':=\beta_2-\tfrac{c_2}{24} \tag{6.38}$$

chosen such that the total magnetic charge $P=S_1-S_2$ is fixed at some large value inside the Kähler cone. The intersection product is $\langle\Gamma_2,\Gamma_1\rangle = P^3/6-P\cdot(\beta_1'+\beta_2')+n_1-n_2 = I_P-P\cdot(\beta_1+\beta_2)+n_1-n_2$. In the large radius limit $J\to\infty$, the stability condition (3.23) simply amounts to

$$\langle\Gamma_2,\Gamma_1\rangle = \frac{P^3}{6}-P\cdot(\beta_1'+\beta_2')+n_1-n_2>0. \tag{6.39}$$

In the limit $P\to\infty$, n_i, β_i fixed, this is automatically satisfied. Recall however that this is only a necessary, not a sufficient condition for existence.

When $n_i,\beta_i'=0$, we have essentially the extremal case $\tilde n=\tilde\beta=0$ in the class of examples studied section 3.4. Indeed we saw there that in this case these always exist as

[43] Recall that the case $r=0$ was excluded by (3.62).

bound states, and that this remains true for small perturbations away from $\tilde{n} = \tilde{\beta} = 0$ as long as the charges Γ_1 and Γ_2 support BPS states (see fig. 3).

Let us be more precise. To establish the existence of a D4-D2-D0 bound state at large radius, it is sufficient to establish this at a conveniently chosen value of the B-field, since we know the infinite radius limit of the D4-D2-D0 spectrum is invariant under shifts of B.

We will take this value to be $\tilde{B} = 0$ after making the uniformizing change of variables $B \to \tilde{B}$ (3.3). In the case at hand

$$B = \frac{S_1 + S_2}{2} + D_A D^{AB} \Delta\beta_B + \tilde{B}, \qquad \Delta\beta \equiv \beta_2 - \beta_1 \tag{6.40}$$

which puts the total central charge in the form (3.4). Using this it is straightforward to show that the attractor flow of the total charge Γ starting at $\tilde{B} = 0$ will remain at $\tilde{B} = 0$ and run straight down till it crashes on a zero of the central charge at $J_0 = \sqrt{2\widehat{q}_0/P^3}\, P$, where

$$\widehat{q}_0 := q_0 - \frac{1}{2} D^{AB} q_A q_B = \frac{P^3}{24} - \frac{1}{2} P(\beta_1' + \beta_2') + n_1 - n_2 - \frac{1}{2}(\Delta\beta)^2 > 0, \tag{6.41}$$

and $(\Delta\beta)^2$ is defined with the D^{AB} metric. If we moreover take the initial point of the flow at $J_\infty = y_\infty P$, the flow will simply be given by $J = y\, P$, where y runs down from y_∞ to $y_0 = \sqrt{2\widehat{q}_0/P^3}$. Hence our choice of $\tilde{B} = 0$ corresponds to a line to which attractor flows coming in from large radius converge, thus making it a particularly natural choice to make.

In order for the first split $\Gamma \to \Gamma_1 + \Gamma_2$ of the flow tree to exist, a wall of marginal stability must be met before the attractor flow hits y_0. Since the total central charge is real along the flow, this wall must be at a solution y of $\mathrm{Im} Z(\Gamma_1)|_y = \mathrm{Im} Z(\Gamma_2)|_y = 0$. Thus the split point is given by

$$t_{\rm ms} = (B + iJ)_{\rm ms} = \frac{S_1 + S_2}{2} + D_A D^{AB} \Delta\beta_B + i y_{\rm ms} P. \tag{6.42}$$

where we choose the unique positive root:

$$y_{\rm ms} = \frac{1}{\sqrt{P^3}} \sqrt{\frac{3P^3}{4} - 3P(\beta_1' + \beta_2') + 3(\Delta\beta)^2}. \tag{6.43}$$

Note that the argument of the square root is positive. To have $y_{\rm ms} > y_0$, we thus need

$$\frac{3P^3}{8} - \frac{3}{2} P(\beta_1' + \beta_2') + \frac{3}{2}(\Delta\beta)^2 > \widehat{q}_0. \tag{6.44}$$

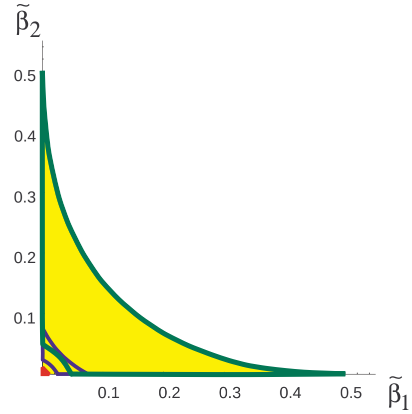
Note that this again automatically satisfied when $P \to \infty$ at fixed n_i, β_i.

This condition is still not quite enough however, since it is not enough for the $Z(\Gamma_i)$ to be real to have a true marginal stability wall at $y = y_{\rm ms}$ — they must have the same sign as well. In fact this sign must be positive since the total central charge Z is positive in the limit $y \to \infty$ and remains so till it hits zero. This gives the somewhat complicated conditions

$$Z_1|_{\rm ms} = \frac{P^3}{6} + \frac{3}{2}\frac{P\Delta\beta}{P^3}\left(P(\beta_1'+\beta_2') - (\Delta\beta)^2\right)$$
$$-P\beta_2' + (\Delta\beta)^2 - \frac{1}{2}(\beta_1'+\beta_2')\Delta\beta + \frac{(\Delta\beta)^3}{6} - n_1 > 0 \tag{6.45}$$
$$Z_2|_{\rm ms} = \frac{P^3}{6} - \frac{3}{2}\frac{P\Delta\beta}{P^3}\left(P(\beta_1'+\beta_2') - (\Delta\beta)^2\right)$$
$$-P\beta_1' + (\Delta\beta)^2 + \frac{1}{2}(\beta_1'+\beta_2')\Delta\beta - \frac{(\Delta\beta)^3}{6} + n_2 > 0. \tag{6.46}$$

Here $(\Delta\beta)^3 := D_{ABC}(\Delta\beta)^A(\Delta\beta)^B(\Delta\beta)^C$, with $(\Delta\beta)^A := D^{AB}(\Delta\beta)_B$. Again, these conditions are automatically satisfied when $P \to \infty$ at fixed β_i, n_i.

In summary, when $\widehat{q}_0 > 0$ the conditions for the split flow $\Gamma \to \Gamma_1 + \Gamma_2$ to exist are given by the inequalities (6.44), (6.45) and (6.46).[44] To correspond to an actual BPS bound state, we furthermore need that Γ_1 and Γ_2 each support BPS states at $y = y_{\rm ms}$. This is straightforward if Γ_1 and Γ_2 are realized as single attractor flows, but becomes again nontrivial when these charges themselves correspond to split flows: this is one of the main technical difficulties we face.



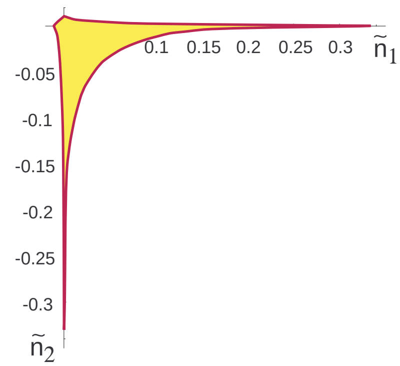


Figure 17: Left: The yellow shaded area is the projection of $\mathcal{S}[0,1]$ into the $(\tilde\beta_1, \tilde\beta_2)$-plane. The green outline is the projection of $\mathcal{S}[.9,1]$, the (smaller) blue one of $\mathcal{S}[.4,.5]$ and the (smallest) red one of $\mathcal{S}[0,.1]$. **Right:** Projection of $\mathcal{S}[0,1]$ into the $(\tilde n_1, \tilde n_2)$-plane.

To get a feeling for the implications of these conditions let us consider the simplest example: We take a CY with one Kähler modulus (for example the quintic CY) and suppose that Γ_1 and Γ_2 support single center attractor flows. Parametrizing $\beta_i' =: \tilde\beta_i P^2$, $n_i\omega =: \tilde n_i P^3$, $\widehat{q}_0 =: (1-\eta)P^3/24$, the P-dependence scales out of all inequalities (6.39)-(6.46), while the condition (3.19) for existence of the regular attractor points for Γ_i becomes

$$\tilde\beta_i \geq 0, \qquad 8\tilde\beta_i^3 - 9\tilde n_i^2 \geq 0. \tag{6.47}$$

[44] It can be checked easily that when $\widehat{q}_0 > 0$, (6.44) actually implies (6.39). Given the other two inequalities, one can also replace $\widehat{q}_0$ by 0 on the right hand side of (6.44), since existence of $y_{\rm ms}$ and positivity of $\mathrm{Re}\, Z$ there imply that $y_{\rm ms} > y_0$, as $\mathrm{Re}\, Z$ is positive at $y = \infty$ and changes sign at $y = y_0$.

Since the system of inequalities is rather complicated we scanned the solution spaces

$$\mathcal{S}[a,b] := \{(\tilde\beta_1,\tilde\beta_2,\tilde n_1,\tilde n_2)|\ \text{2-centered solution exists with } a \le 1-\eta \le b\} \tag{6.48}$$

numerically for various intervals $[a,b] \subseteq [0,1]$. Fig. 17 shows the corresponding projections to the $(\tilde\beta_1,\tilde\beta_2)$- and $(\tilde n_1,\tilde n_2)$-planes.

One thing that transpires from this analysis which is not immediately obvious from the inequalities, although expected physically, is that, taking into account charge quantization, the solution space is finite. It is furthermore clear from the plots that the solution space does not factorize, in the sense that the choice of $(\tilde\beta_1,\tilde n_1)$ influences the stability domain of $(\tilde\beta_2,\tilde n_2)$. Another distinct feature is the correlation between the size of $(\tilde\beta_i,\tilde n_i)$ and η: the more polar the state is, i.e. the closer η approaches 0, the smaller $\tilde\beta_i$ and $\tilde n_i$ are forced to be. For $\tilde\beta_i,\tilde n_i \ll 1$ this is not hard to deduce analytically in the case at hand. When $\tilde\beta_i,\tilde n_i \ll 1$, all the required inequalities are automatically satisfied, except (6.47), which remains nontrivial. The relation between η and the $(\tilde\beta_i,\tilde n_i)$ is given by

$$\eta = \frac{1}{2}(\tilde\beta_1+\tilde\beta_2) - \tilde n_1 + \tilde n_2 + \frac{1}{2}(\tilde\beta_1-\tilde\beta_2)^2. \tag{6.49}$$

When $\tilde\beta_i \ll 1$, the term quadratic in the $\tilde\beta_i$ is negligible compared the term linear in the $\tilde\beta_i$, and $\tilde n_1 - \tilde n_2$ as well because of (6.47). Thus we get the simple relation $\tilde\beta_1+\tilde\beta_2 = 2\eta$, $\tilde\beta_i > 0$, $3\tilde n_i < (2\tilde\beta_i)^{3/2}$, making it obvious that $\tilde\beta_i$ and $\tilde n_i$ get smaller when $\widehat{q}_0$ approaches its maximum, i.e. $\eta \to 0$.[45]

In section 6.2.2 we will conjecture that the behavior exhibited in this example persists in the general case as well, namely, that the most polar states correspond to β_i, n_i which are in some sense small compared to the scales set by P.

We are thus interested in charges in which the β_i, n_i are "small" compared to the scales set by P. In the language of [17, 18], these are "dilute gas" states, which in our setup can be thought of microscopically as D2-D0 branes sparsely floating around inside the D6 and the anti-D6 branes. Let us now make this notion of dilute gas more precise.

Define the scale of P by

$$|P| := (P^3)^{1/3}. \tag{6.50}$$

We will take it to be large, and in the OSV conjecture it will scale to infinity. Next, define a set of "small" (β,n) as follows[46]

$$\mathcal{C}(P,\epsilon) := \{(\beta,n)\,|\,\beta \text{ effective},\ \beta\cdot P < \epsilon_{|P|}\,|P|^3,\ |n| < \epsilon_{|P|}\,|P|^3\}. \tag{6.51}$$

Note that because P is very ample and β effective, we have in components with respect to a basis of the Kähler cone that $\beta_A \ge 0$, $P^A > 0$, so the bound on β implies for each component

$$\beta_A < \epsilon\,\frac{|P|}{P^A}\,|P|^2 \sim \mathcal{O}(\epsilon|P|^2), \tag{6.52}$$

[45] Actually the maximal value of $\widehat{q}_0$ is $(P^3 + c_2\cdot P)/24$, but in the $P\to\infty$ supergravity regime we have in mind here, the linear correction is negligible.

[46] By $\epsilon_{|P|}$ we mean to indicate that ϵ can be taken to depend on $|P|$, e.g. $\epsilon_{|P|} \sim |P|^{-\xi}$, for $|P|\to\infty$. We will usually just write ϵ though.

where we used that $\frac{|P|}{P^A} \sim \mathcal{O}(|P|^0)$ when we scale up P uniformly. Using this, it is clear that for sufficiently small ϵ we have

$$(\beta_1, n_1) \ \& \ (\beta_2, n_2) \in \mathcal{C}(P, \epsilon) \Rightarrow (6.44), (6.45) \text{ and } (6.46) \text{ are satisfied.} \tag{6.53}$$

and hence the split $\Gamma \to \Gamma_1 + \Gamma_2$ exists.

6.2.2 The extreme polar state conjecture

In the previous section we have examined a particular class of examples in which a D4-D2-D0 BPS state is accounted for as a D6-antiD6 split state. We are particularly, interested in polar states. As we will see, the "more polar" a state is - that is, the larger the value of $\hat{q}_0$ - the more important is the contribution of that polar state to the OSV formula. Clearly, the polar states analyzed in the previous section do not account for all polar states, since they only involve D6 branes with $r = 1$. In this section we will state a conjecture which claims that nevertheless, if we restrict attention to sufficiently polar states, then the examples of the previous section are indeed the most general examples. We will give some evidence for this conjecture.

We know that any polar state splits into charges

$$\Gamma_1 = re^{S_1}(1 - \beta_1 + n_1\,\omega), \quad \Gamma_2 = -re^{S_2}(1 - \beta_2 + n_2\,\omega), \tag{6.54}$$

where $r(S_1 - S_2) = P$. Subsequent splits can also occur, but here we are only interested in the first split. Splits in charges with zero D6 charge were excluded by (3.62).

For such a split, we have

$$\widehat{q}_0 = r\left(\frac{\widehat{P}^3}{24} - \frac{1}{2}\widehat{P}\cdot(\beta_1 + \beta_2) + n_1 - n_2 - \frac{1}{2}(\Delta\beta)^2\right), \qquad \widehat{P} := \frac{P}{r} \tag{6.55}$$

where $(\Delta\beta)^2 = (D_{ABC}\widehat{P}^C)^{-1}\,(\beta_{1,A} - \beta_{2,A})(\beta_{1,B} - \beta_{2,B})$.

We can introduce a measure of the degree of polarity of a D4-D2-D0 BPS state by defining

$$\eta := \frac{(\hat{q}_0)_{\max} - \hat{q}_0}{(\hat{q}_0)_{\max}}, \tag{6.56}$$

where $(\hat{q}_0)_{\max} = \frac{P^3 + c_2 P}{24}$. Throughout, we will think of $|P|$ as being very large, though finite. Therefore to good approximation, we can drop c_2 corrections, which for simplicity we will do in what follows. We will define *extreme polar states* as those for which $\eta \ll 1$. Then, we conjecture that for sufficiently small $\eta < 1$ there exists an $\epsilon(\eta)$ sufficiently small so that the restricted class of D6-anti-D6 bound states with charges drawn from the set $\mathcal{C}(P; \epsilon(\eta))$ defined in (6.51) indeed account for all such extremely polar D4-D2-D0 BPS states.

It follows easily from the above formulae that $r = 1$ D6-anti-D6 bound states with $(\beta_i, n_i) \in \mathcal{C}(P, \epsilon)$ have $\eta < \mathcal{O}(\epsilon)$, and hence are extreme polar for small ϵ. What we would like to know is the converse, namely that extreme polar states are *only* realized by splits with $r = 1$ and small β_i, n_i. More precisely, we would like to prove the:

Extreme polar state conjecture:

a.) For any $\eta_* \ll 1$, there exists an $\epsilon(\eta_*) \ll 1$ such that every D4-D2-D0 BPS state with $\eta < \eta_*$ corresponds to a split (Γ_1, Γ_2) as in (6.37)-(6.38), with $(\beta_i, n_i) \in \mathcal{C}(P, \epsilon(\eta_*))$ as defined in (6.51).

b.) Moreover, there is a P-independent constant, μ so that we may take $\epsilon = \mu\eta_*$.

Some heuristic intuition for the absence of "large" (β_i, n_i) contributions to extreme polar state realizations is (i) the complexity and entropy of D4-D2-D0 states increases with η, and (ii) the complexity and entropy of D6-D4-D2-D0 states increases with the scale of (β_i, n_i), so for (β_i, n_i) too large, we would get a contribution with too much entropy. In the one modulus examples in section 6.2, in particular the discussion around (6.49), we saw this proportionality relation between η and the scale of β_i, n_i explicitly.

The absence of $r > 1$ splits from the extreme polar spectrum is perhaps more surprising at first sight, but becomes less so when one notes that for $\beta_i = 0$, $n_i = 0$, $\hat{q}_0 = P^3/(24r^2)$, so $\eta = 1 - 1/r^2 \geq 3/4$ for $r \geq 2$. We also conjecture that the latter is the maximal possible value for $\hat{q}_0$ at any given value of r, reached iff $\beta_i = 0$, $n_i = 0$, i.e. for a bound state of a pure $U(1)$ fluxed stack of D6-branes and a stack of anti-D6 branes. The truth of this latter conjecture is not necessary for our derivation of the OSV conjecture.

Unfortunately, we have not been able to find a full, general proof of the extreme polar state conjecture. Within the class of $r = 1$ bound states, the main problem is to find suitable bounds on the positive contributions to $\hat{q}_0$ in (6.41), such that cancelations between "large" values of the β_i, n_i are avoided. It seems reasonable that such large cancelations are absent, since just a slight change of such canceling large parameters would transform an extreme polar state ($0 < \eta \ll 1$) into a super-polar state ($\eta < 0$), which we know are absent. However when looking in more detail, one finds that the bounds come from many different existence criteria, and all of them, including complicated inequalities like (6.45) and (6.46) as well as various stability conditions for constituent D6-D2-D0 bound states must be taken into account to prevent such cancelations from happening.

Let us nevertheless have a closer look at the conjecture. First note that $\mathcal{C}(P, \epsilon)$ is defined such that at fixed ϵ and ignoring charge quantization, $(\beta, n) \in \mathcal{C}(P, \epsilon)$ iff $(\lambda^2\beta, \lambda^3 n) \in \mathcal{C}(\lambda P, \epsilon)$, i.e. it respects the scaling symmetry (3.27), as it should for the extreme polar state conjecture to make sense (since this relates ϵ to η_* which similarly is invariant under the above rescalings).

It is not hard to show that the conjecture is indeed nontrivially true for the important special case of polar states splitting in two single centered black holes with charges

$$\Gamma_1 = re^{S_1}(1 - \beta_1 + n_1\,\omega), \qquad \Gamma_2 = -re^{S_2}(1 - \beta_2 + n_2\,\omega), \tag{6.57}$$

where we take $\beta_1 = \beta_2 =: \beta$, such that we don't have to worry about possible positive contributions from the $(\Delta\beta)^2$ term in (6.41). For simplicity we will also take $-n_1 = n_2 =: n$, but this can be easily generalized to $n_1 \neq n_2$.

We then get

$$\widehat{q}_0 = r\left(\frac{\widehat{P}^3}{24} - \widehat{P}\cdot\beta - 2n\right), \qquad \widehat{P} := \frac{P}{r}, \tag{6.58}$$

and

$$\eta = (1 - \frac{1}{r^2}) + \frac{24}{r^2} \frac{\widehat{P} \cdot \beta + 2n}{\widehat{P}^3}. \tag{6.59}$$

Furthermore the split existence conditions (6.44), (6.45) and (6.46) reduce simply to[47]

$$\frac{\widehat{P}^3}{6} - \widehat{P} \cdot \beta + n > 0, \tag{6.60}$$

while for the black hole constituents to exist, (3.19) must be satisfied, i.e.

$$8(Y^3)^2 - 9\,n^2 \geq 0, \quad Y^2 := \beta, \quad Y > 0. \tag{6.61}$$

Now a general inequality [48] for any triplet of divisors X, Y, Z inside the Kähler cone is [99] (Theorem 1.6.1) $X^3\,Y^3\,Z^3 \leq (X \cdot Y \cdot Z)^3$, so in particular we have $(Y^3)^2 \widehat{P}^3 \leq (Y^2 \widehat{P})^3 = (\beta \cdot \widehat{P})^3$, and from this

$$|n| \leq \sqrt{\frac{8(\beta \cdot \widehat{P})^3}{9\,\widehat{P}^3}}. \tag{6.62}$$

Denoting $\widehat{P} \cdot \beta =: \tilde{\beta} \widehat{P}^3$, $n =: \tilde{n} \widehat{P}^3$, the above expressions become

$$\eta = 1 - \frac{1}{r^2} + \frac{24}{r^2}(\tilde{\beta} + 2\tilde{n}), \qquad \frac{1}{6} - \tilde{\beta} + \tilde{n} > 0, \qquad |\tilde{n}| \leq \sqrt{\frac{8\tilde{\beta}^3}{9}}. \tag{6.63}$$

Note that the inequalities are of exactly the same form as the existence conditions in our class of examples studied in section 3.4. From the analysis there, we can therefore immediately conclude that $\tilde{\beta} + 2\tilde{n} \geq 0$. This implies $\eta \geq 1 - \frac{1}{r^2}$, which immediately excludes all $r > 1$ configurations, since we are considering extreme polar states here, which by definition have $\eta \ll 1$. For $r = 1$, we have furthermore $\eta = 1 - 24\,\nu$ in the notation of section 3.4, and lines of constant η in fig. 3 are given by translations of the purple dotted line, with $\eta = 1$ corresponding to the original line and $\eta = 0$ to its translation to the left such that it contains the origin $(\tilde{\beta}, \tilde{n}) = (0, 0)$. It is then clear from the plot that taking η smaller and smaller will also cause $\tilde{\beta}$ and $\tilde{n}$ to become smaller and smaller.

A precise bound is easily obtained by using $\tilde{\beta} \leq \frac{1}{8}$ (as can be read off from fig. 3) together with $\eta = 24(\tilde{\beta} + 2\tilde{n}) \geq 24(\tilde{\beta} - \frac{4\sqrt{2}}{3}\tilde{\beta}^{3/2})$, which gives $\tilde{\beta} \leq \frac{\eta}{8}$, $|\tilde{n}| \leq \frac{\eta^{3/2}}{24}$. This shows that all extreme polar 2-centered configurations with charges given by (6.57) have $r = 1$ and have $(\beta_i, n_i) \in \mathcal{C}(P, \epsilon)$, where we can take $\epsilon = \frac{\eta}{8}$, thus establishing the extreme polar state conjecture for this case. [49]

It is relatively straightforward to extend this proof to the case where we add D2-D0 halos around the black hole centers while still keeping $\beta_1 = \beta_2$ and $-n_1 = n_2 = n$. The conditions for having a split point remain unchanged, since these do not care about

[47] Although we derived these stability inequalities only for the case $r = 1$, this is trivially extended to general r by using the uniform scaling symmetry (3.26), under which $r \to \mu r$, $r(S_1 - S_2) = P \to \mu P$, and β, n are invariant.

[48] Recently used in [100].

[49] Recently, E. Andriyash has extended the argument to allow $\beta_1 \neq \beta_2$ and $n_1 \neq n_2$.

the composition of Γ_1 and Γ_2. Evaluating (6.14) leads to the simple stability condition $n_h(1+6\tilde{n}-6\tilde{\beta})>0$ for halos around the second center, and the opposite inequality for the first center. Now the quantity within brackets is actually positive because of the split conditions given above, so the halo stability condition simply becomes $n_h>0$ for Γ_2 and $n_h<0$ for Γ_1. This means that the only effect of adding these halos will be to remove the red curved boundary on the right in fig. 3, extending the stable (=yellow shaded) region to the downwards sloping blue line on the far right. It is then again clear from the plot that the extreme polar state conjecture holds in this case.

Things become more complicated when we allow more general multicentered core configurations (such as those described in section 6.1.3), or core configurations with $\beta_1\neq\beta_2$. We analyzed in depth a number of examples, and always found the extreme polar state conjecture to be true. However the detailed arguments we found vary from case to case, tend to be messy, and are not particularly illuminating as to why the conjecture should be true in general, so we will not report the details here.

On the other hand, it is possible to give a more general (albeit incomplete) scaling argument for why only $r=1$ splits contribute to the extreme polar states. Say we start with a multicentered BPS configuration with initial split having $r\geq 2$. Then we can produce from this a multicentered BPS configuration with the same total P but $r=1$ by applying the scaling symmetries of section 3.3 with $\mu=r^{-1}$, $\lambda=r$. This scales all $(p^0,p,q,q_0)\to(p^0/r,p,rq,r^2q_0)$ and in particular $\hat{q}_0\to r^2\hat{q}_0$. However we know that the maximal possible $\hat{q}_0$ equals $(\hat{q}_0)_{\rm max}=P^3/24$, so in particular we have $r^2\hat{q}_0<(\hat{q}_0)_{\rm max}$, hence for our original configuration $\eta>1-\frac{1}{r^2}\geq\frac{3}{4}$, implying it is not extreme polar.

Regrettably, this argument has a flaw: if some of the centers have D6-charge p^0 not divisible by r, the rescaled configuration violates charge quantization and is therefore unphysical, so we cannot use the physical bound on $\hat{q}_0$ (if we allowed fractional p^0 we could produce super-polar states, so we should be strict as far as D6-charge quantization is concerned here). One could therefore worry that, for example, by splitting up the $p^0=\pm r$ centers of the class of 2-centered solutions analyzed below (6.57), we could make $\hat{q}_0$ greater than the bound derived there, or equivalently η smaller than $1-1/r^2$.

A full analytical analysis of such multicentered configurations with smaller D6-charges becomes rather cumbersome. Instead we performed a numerical analysis of the four centered $\mathrm{D6}-\mathrm{D6}-\overline{\mathrm{D6}}-\overline{\mathrm{D6}}$, $r=2$ case, searching through ensembles of flow trees by a simple adaptive random walk optimization method, trying to maximize $\hat{q}_0$. The results we obtained are fully consistent with the extreme polar state conjecture. We refer to appendix D for more details.

In what follows we will assume the extreme polar state conjecture is true.

6.3 The dilute gas D6-anti-D6 partition function

6.3.1 Definition and factorization

Let us define the following generating function

$$\mathcal{Z}^{\epsilon}_{D6-\overline{D6}}(u,v,w):=\sum_{\Gamma_1,\Gamma_2}\Omega(\Gamma_1)_{\rm ms}\,\Omega(\Gamma_2)_{\rm ms}\,u^{q_0}\,v^{Q}\,w^{\langle\Gamma_2,\Gamma_1\rangle},\tag{6.64}$$

where Γ_1, Γ_2 are parametrized as in (6.37)-(6.38) with P (but not S) fixed and with $(\beta_i, n_i) \in \mathcal{C}(P, \epsilon)$ as defined in (6.51). Here q_0 and Q are the total D0- resp. D2-brane charges (recall eqs. (4.22)):

$$Q = \beta_2 - \beta_1 + PS, \qquad S := \frac{S_1 + S_2}{2} \tag{6.65}$$

$$q_0 = \frac{P^3 + c_2 P}{24} + \frac{1}{2} PS^2 - S\beta_1 - \frac{P}{2}\beta_1 + n_1 + S\beta_2 - \frac{P}{2}\beta_2 - n_2, \tag{6.66}$$

and the subscript "ms" as before means that the indices have to be evaluated at the split point of the attractor flow $\Gamma \to \Gamma_1 + \Gamma_2$, as given by (6.42).

Two remarks on this definition are in order:

1. Note that

$$\frac{\partial}{\partial w} \mathcal{Z}^{\epsilon}_{D6-\overline{D6}}(u, v, w)\Big|_{w=-1} = \sum_{\Gamma_1, \Gamma_2} (-1)^{\langle \Gamma_1, \Gamma_2 \rangle - 1} |\langle \Gamma_1, \Gamma_2 \rangle| \, \Omega(\Gamma_1)_{\rm ms} \, \Omega(\Gamma_2)_{\rm ms} \, u^{q_0} \, v^Q, \tag{6.67}$$

so, comparing to (5.4), we see that the coefficients of this derivative count the indices of our D6-anti-D6 BPS bound states for given total charge (Q, q_0).

2. The sum over β_i, n_i is (most likely) a finite sum, but the sum over S is definitely an infinite sum. The sum on S is always divergent because the quadratic form defined by PS^2 (which appears through q_0) has signature $(1, h-1)$. However (6.64) does make good sense as a formal power series, in the sense that only a finite number of terms contributes to the coefficient of any monomial $u^n v^\beta w^\ell$. This is true simply because the map $S \to PS$ is invertible.

We now aim to write (6.64) as a sum over S of factorized expressions depending only on (β_1, n_1) and (β_2, n_2), respectively. To this end let us evaluate the degeneracy factors Ω. We are instructed by (5.4) to compute $\Omega(\Gamma_i)$ evaluated at the split point (6.42). It is convenient at this point to use the gauge invariance under shifting the B-field to say:

$$\Omega(\Gamma_1; t_{\rm ms}) = \Omega(\Gamma(\beta_1, n_1); t^1_{\rm ms}) \tag{6.68}$$

$$\Omega(\Gamma_2; t_{\rm ms}) = \Omega(\Gamma(\beta_2, n_2); t^2_{\rm ms}). \tag{6.69}$$

If $\Gamma_1, \Gamma_2 \in \mathcal{C}(P, \epsilon)$ then note that for the gauge invariant quantity $\mathcal{F}_i = S_i - B$ we have, up to $\mathcal{O}(\epsilon|P|)$ corrections, $\mathcal{F}_1 = \frac{P}{2}$, $\mathcal{F}_2 = -\frac{P}{2}$. Accounting for the imaginary part we have:

$$t^1_{\rm ms} = -\frac{1}{2}P + D_A D^{AB} \Delta\beta_B + i y_{\rm ms} P = e^{2\pi i/3} P + \mathcal{O}(\epsilon|P|) \tag{6.70}$$

$$t^2_{\rm ms} = \frac{1}{2}P + D_A D^{AB} \Delta\beta_B + i y_{\rm ms} P = e^{\pi i/3} P + \mathcal{O}(\epsilon|P|). \tag{6.71}$$

Thus the degeneracies are counted by the generating function (6.16) evaluated at $B + iJ = (\pm\frac{1}{2} + i\frac{\sqrt{3}}{2})P + \mathcal{O}(\epsilon|P|)$, so up to ϵ corrections the degeneracies are indeed independent of each other!

Using (6.39), (6.65) and (6.66), we can write

$$\begin{aligned}
\mathcal{Z}^{\epsilon}_{D6-\overline{D6}}(u,v,w) &= \sum_{S,\beta_i,n_i} \Omega(\Gamma(\beta_1,n_1);t^1_{\rm ms})\,\Omega(\Gamma(\beta_2,n_2);t^2_{\rm ms}) \\
&\qquad \times w^{I_P - P\beta_1 - P\beta_2 + n_1 - n_2} \\
&\qquad \times u^{\frac{P^3+c_2P}{24}+\frac{1}{2}PS^2 - S\beta_1 - \frac{P}{2}\beta_1 + n_1 + S\beta_2 - \frac{P}{2}\beta_2 - n_2}\, v^{\beta_2-\beta_1+PS} \qquad (6.72)\\
&= u^{\frac{P^3+c_2P}{24}}\, w^{I_P} \sum_S u^{\frac{1}{2}PS^2} v^{PS} \sum_{\beta_1,n_1}\sum_{\beta_2,n_2} \\
&\quad \times \Omega(\Gamma(\beta_1,n_1);t^1_{\rm ms})\,(wu)^{n_1}\,(w^{-P}u^{-S-\frac{P}{2}}v^{-1})^{\beta_1} \\
&\quad \times \Omega(\Gamma(\beta_2,n_2));t^2_{\rm ms})\,(wu)^{-n_2}\,(w^{-P}u^{S-\frac{P}{2}}v)^{\beta_2} \qquad (6.73)
\end{aligned}$$

where the sums are over $S \in \frac{P}{2} + H^2(X,\mathbb{Z})$, $(\beta_i,n_i) \in \mathcal{C}(P,\epsilon)$, and $t^{1,2}_{\rm ms}$ refers to the shifted marginal stability points (6.70),(6.71).

Note that the sum is almost factorized. For the next step we would like to rewrite (6.73) as a sum of products of DT partition functions. We will eventually achieve this, under suitable conditions, in equation (6.94) below, but first, in view of the identifications (6.21)-(6.24), we need to compare

$$\Omega(\Gamma(\beta_1,n_1);t^1_{\rm ms}) \qquad (6.74)$$

with

$$\lim_{z\to\mathcal{L}^\pm} \Omega(\Gamma(\beta_1,n_1);zP). \qquad (6.75)$$

where $\lim_{z\to\mathcal{L}^\pm}$ indicates that z goes to infinity in the left-half plane as explained just above eq. (6.21). Similarly we need to compare

$$\Omega(\Gamma(\beta_2,n_2);t^2_{\rm ms}) \qquad (6.76)$$

with

$$\lim_{z\to\mathcal{R}^\pm} \Omega(\Gamma(\beta_2,n_2);zP). \qquad (6.77)$$

The relation between (6.74) and (6.75) and between (6.76) and (6.77) is not at all trivial, and in fact they *will* differ in general, due to jumps at marginal stability. In the next section we compare these two degeneracies.

Note that, roughly, the M-theory equivalent to this is that there can be BPS states which exist in Taub-NUT when the Taub-NUT radius is taken to infinity, but not necessarily at arbitrary finite radii, and similarly they do not necessarily exist when the Taub-NUT is combined with an anti-Taub-NUT to produce the finite size $AdS_3 \times S^2$ setup of [17, 18]. In particular this implies that it is not true that BPS states in these finite size cases are exactly counted by the GV / DT partition function (which counts BPS states in infinite radius Taub-NUT). Figuring out to what extent the spectrum is truncated is a difficult problem, and the absence of a systematic way to do this is what prevented [17, 18] from arriving at any error estimates in their derivation of the OSV conjecture.

Happily, the tools we have developed in this paper are exactly designed to do this, so let us now turn to this analysis.

6.3.2 Harmless Halos and Catastrophic Cores

We now focus on the difference

$$\Delta\Omega(\beta_1, n_1; \beta_2, n_2) = \Omega(\Gamma(\beta_1, n_1); t^1_{\rm ms}(\beta_1, n_1; \beta_2, n_2)) - \lim_{z\to\mathcal{L}^\pm} \Omega(\Gamma(\beta_1, n_1); zP). \tag{6.78}$$

Nonzero contributions to $\Delta\Omega$ will lead to corrections in the OSV-like relation we wish to derive. The reason $\Delta\Omega$ can be nonzero is that there can be BPS states of charge $\Gamma(\beta_1, n_1)$ which are stable at $t^1_{\rm ms}$ but unstable at infinity and vice versa. States which make a nonzero contribution to (6.78) will be called *swing states*.

To be more precise, we will call a D6-D2-D0 BPS state of charge $\Gamma(\beta_1, n_1)$ a *swing state* if $(\beta_1, n_1) \in \mathcal{C}(P, \epsilon)$ and there exists $(\beta_2, n_2) \in \mathcal{C}(P, \epsilon)$ such that either the state is contained in $\mathcal{H}(\Gamma(\beta_1, n_1); t)$ with $t = zP$, $z \to \mathcal{L}^\pm$, but decays along the way to $t = t^1_{\rm ms}(\beta_1, n_1; \beta_2, n_2)$, or vice versa, i.e. it exists at $t = t^1_{\rm ms}$ but not at $t \to \mathcal{L}^\pm P$. Recall that $t^1_{\rm ms} = e^{2i\pi/3}P$ up to order ϵ corrections, so the definition basically says that a D6-D2-D0 state is a swing state when it exists at infinity but not in an order ϵ neighborhood of $t = e^{2i\pi/3}P$, or vice versa.

A very useful simplification in the analysis of swing states arises when we recall that we are evaluating the stability condition at a special point, $t^1_{\rm ms}$. Since $Z(\Gamma_1; t_{\rm ms}) > 0$ the stability condition for

$$\Gamma(\beta_1, n_1) \to \tilde\Gamma + \tilde\Gamma' \tag{6.79}$$

simplifies to:

$$\langle \tilde\Gamma, \Gamma(\beta_1, n_1)\rangle \, {\rm Im}\, Z(\tilde\Gamma; t^1_{\rm ms}) > 0 \tag{6.80}$$

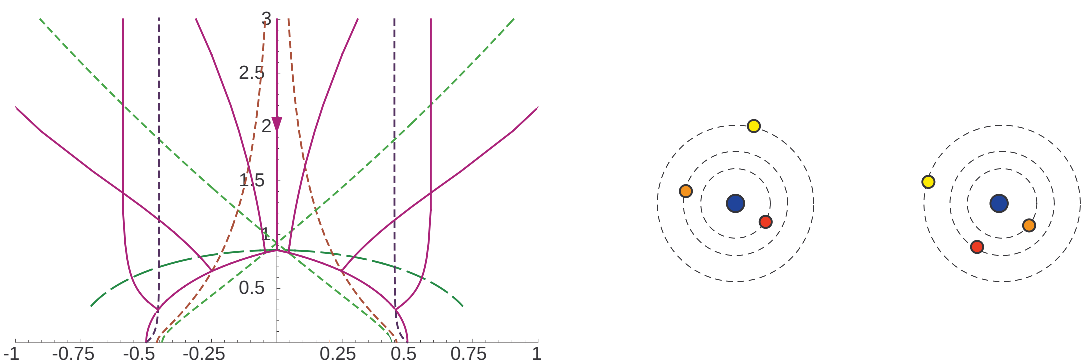


Figure 18: Left: Flow tree corresponding to bound state of a fluxed D6 with three halos, and its conjugate. The charges for the Γ_1 half of the tree are $\Gamma_{1,c} = e^{P/2}$, $\Gamma_{1,h,i} = e^{P/2}(\tilde q_{2,i} P^2 + \tilde q_{0,i} P^3\,\omega)$ with $\tilde q_2 = (-10^{-3}, -10^{-3}, -10^{-3})$ and $\tilde q_0 = (-10^{-4}, -10^{-3}, -10^{-2})$. The larger the D0-charge (at fixed D2), the sooner the D2D0 particles split off in the tree (so the first to split off is $\Gamma_{1,h,3}$). The charges for the Γ_2 half are obtained by taking the conjugates $\Gamma \to -\Gamma^*$. The dotted lines are the MS lines corresponding to the various splits. **Right:** Sketch of a corresponding multicentered configuration.

We will now show that halos do *not* cause configurations to become swing states. Halo states have lines of MS going to infinity and hence one might imagine these wall crossings would make a significant contribution to the difference $\Delta\Omega$. But that turns out not to be the case: The marginal stability curves for $\Gamma(\beta_1, n_1) \to \Gamma(\beta_c, n_c) + \Gamma_h$ with $\Gamma_h = (-\beta_h + n_h \omega)$ bend over quickly from a line of slope $-\sqrt{3}$ to a vertical line when $\beta_h \cdot P \ll n_h$, comfortably keeping $t^1_{\rm ms}$ on their stable side, while when $\beta_h \cdot P \gg n_h$, they come close to $t^1_{\rm ms}$ but still bend over just in time. This behavior can be observed in fig. 18. There is an entirely analogous story for Γ_2 — one just reflects the picture in the y axis using the symmetry (3.29). This special behavior translates into a particularly simple stability condition and (6.80) becomes:

$$\mp n_h \, \beta_h \cdot P > 0 \tag{6.81}$$

where the minus sign is for Γ_1 and the plus sign for Γ_2. Hence for Γ_2 all halos with n_h positive are stable and similarly for Γ_1 all halos with n_h negative are stable.

When the D2-charge of the halo particle vanishes, the situation is more subtle, since in this case the split point lies exactly on the wall of marginal stability for the D0-halo, making the indices ambiguous. To lift the ambiguity, it suffices to take $\tilde{B}$ (defined in (6.40)) slightly different from zero. Then we are essentially in the situation described in section 3.6: stability requires $n_h < 0$, and depending on the chosen sign of $\tilde{B}$, *either* Γ_1 *or* Γ_2 can support D0-halos, but not both at the same time. An important consequence of this is that for counting D6-anti-D6 bound states, we should only include *one* MacMahon factor (6.1) in the dilute gas partition function, and not two as one might have thought naively. We will return to this point below (6.91).

Now let us consider the possibility that there are walls for splitting off other kinds of constituents as we move t_∞ from $t^1_{\rm ms}$ to infinity in the left-half plane. The first observation to make is that if we consider any P-independent finite set of pairs $\{(\beta_1, n_1), (\beta_2, n_2)\}$ then for suffciently large P we have $\Delta\Omega = 0$. We can justify this as follows. According to the split attractor flow conjecture, for any t, $\Gamma(\beta_1, n_1)$ only supports a finite set of split flows. These will begin with some splitting $\Gamma(\beta_1, n_1) = \Gamma_1' + \Gamma_1''$. Now, we know that $t^1_{\rm ms}$ lies to the left of all marginal stability lines where one of Γ_1' or Γ_1'' are halo charges. Thus, we need only worry about the case where Γ_1', Γ_1'' both have magnetic charge. However, in this case the marginal stability lines, which are subvarieties of $\mathrm{Im}(Z(\Gamma_1'; t)\overline{Z(\Gamma_1''; t)}) = 0$ lie in *bounded* regions of moduli space. As long as we consider a set of charges that makes no reference to P, by making P sufficiently large $t^1_{\rm ms}$ (which grows with P) will always be outside the union of the compact regions where splits are allowed. Now, by the argument surrounding (6.81) we can freely take x to $-\infty$ since $t^1_{\rm ms}$ is to the left of all the walls for halo states. A similar argument applies to $\Omega(\Gamma_2)$, where we must take x to $+\infty$.

Regrettably, the above argument is not sufficient for our purposes because the charges in $\mathcal{C}(P; \epsilon)$ can in fact grow with P. As a matter of fact, we will now exhibit a class of examples which *does* give a nonzero contribution to $\Delta\Omega$. That is, we will show that swing states do indeed exist.

To be concrete, we consider candidate bound states of

$$\Gamma_1 = e^{P/2}\Gamma(\beta_1, n_1) \tag{6.82}$$

$$\Gamma_2 = -[e^{P/2}\Gamma(\beta_1, n_1)]^* = -e^{-P/2}\Gamma(\beta_1, -n_1) \tag{6.83}$$

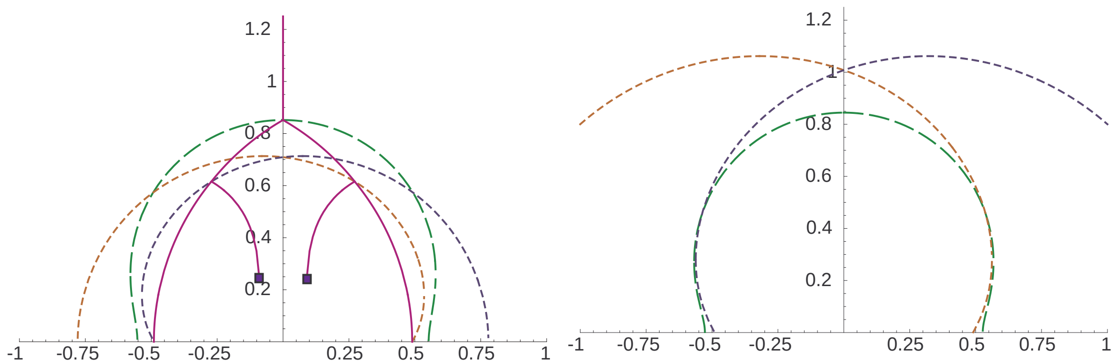


Figure 19: Left: Bound state of two D6-D4 core states as described in the text, with $u = 10^{-3}$, $v = 0.4$, $\tilde{q}_0 = -10^{-4}$, $\frac{\beta_1 \cdot P}{P^3} \approx 10^{-2}$, $\frac{n_1}{P^3} \approx 5 \times 10^{-3}$. **Right:** Failure to form a similar bound state with $v = 0.6$ instead and all other parameters the same. The bound state cannot form because the initial $\Gamma \to \Gamma_1 + \Gamma_2$ split point lies in the unstable region of the constituent core states themselves: the green wide-dashed line is the $\Gamma \to \Gamma_1 + \Gamma_2$ MS line and lies below the short-dotted lines which are the MS lines for the states representing the Γ_i.

where $\Gamma(\beta_1, n_1)$ is realized as one of the core states analyzed in section 6.1.3, equation (6.26). Note that $\beta_2 = \beta_1$, $n_2 = -n_1$.

Recall that the split states for $\Gamma(\beta_1, n_1)$ studied in section 6.1.3 are stable at infinity and unstable within the curve $\mathcal{C}_+$ defined in (6.34). Therefore, such states will contribute to $\Delta\Omega$ if our stability condition is *violated* at t^1_{ms}. Using (6.80) this works out to be the simple condition

$$u + v > 1/2. \tag{6.84}$$

Thus, swing states do exist. An example where this condition is not satisfied (so the bound state does exist) is shown in fig. 19(a), and one where it is satisfied in fig. 19(b) (so the bound state does not exist).

We are now in a position to understand why swing states are potentially problematic. In our example we can compute

$$\hat{q}_0 = \frac{P^3 + c_2 \cdot P}{24} - P(UV + \frac{1}{2}U^2) + [UV^2 + 2U^2V + \frac{2}{3}U^3 + q_0] \tag{6.85}$$

and hence

$$\eta = 24[uv + \frac{1}{2}u^2 - (uv^2 + 2u^2v + \frac{2}{3}u^3) - \tilde{q}_0] \tag{6.86}$$

Note that we can satisfy (6.84) by taking $u \to 0$ while letting $v > 1/2 - u$ be order one. But then $\eta \to 0$ and such states are arbitrarily extreme polar. Looking ahead to the impact on our derivation of the OSV conjecture below, we see that such states will lead to large corrections to the OSV formula arbitrarily close to the leading contribution, invalidating the conjecture. How can we avoid this "coretastrophe"?

Fortunately we can combine a choice of a suitably small ϵ with charge quantization to eliminate the contribution of this particular example of swing states to the dilute gas partition function. Suppose $P = pP_0$ where P_0 is primitive and p will go to infinity. Charge quantization implies $u = m/p$ with m positive and integral. Now, the condition that $(\beta_1, n_1) \in \mathcal{C}(P, \epsilon)$, together with (6.84) and charge quantization implies that

$$\epsilon > uv + \frac{1}{2}u^2 > \frac{1}{2}u(1-u) > \frac{1}{2p}(1-\frac{1}{p}) \tag{6.87}$$

Since we will be taking $p \to \infty$, if we take $\epsilon = \delta/p$ with δ a p-independent constant smaller than 1/2 then (6.87) will eventually be violated, and hence these particular swing states are eliminated from the ensemble defined by $\mathcal{C}(P, \epsilon)$.

The above argument shows that our example of potentially catastrophic [50] swing states can be eliminated by making a suitable (P-dependent) choice of ϵ. Sadly, we have no proof that there are not other swing states which will create problems, so we proceed as follows.

Suppose $\epsilon = \frac{\delta}{|P|^\xi}$ where δ is a P-independent constant. We know that if we choose $\xi = 3$ then the states in $\mathcal{C}(P, \epsilon)$ consist of a finite set of P-independent charges (β_1, n_1). Our argument above shows that for such states indeed $\Delta\Omega = 0$. Unfortunately, this is not enough to prove the OSV conjecture. The reason is that, as we show in equation (6.121) below, the error from the (necessary) restriction to extreme polar states is of order

$$\mathcal{O}\left(\exp[-\frac{\pi}{12\mu}\frac{\epsilon|P|^3}{\phi^0}]\right) \tag{6.88}$$

where μ is the constant $\epsilon = \mu\eta_*$ introduced in the extreme polar state conjecture. On the other hand, the worldsheet instanton effects which make the OSV conjecture nontrivial are of order $\exp[-2\pi\beta \cdot P/\phi^0]$. Therefore, if $\xi > 2$ the states contributing to $\Delta\Omega$ make contributions larger than those of worldsheet instantons, eventually dominating all worldsheet instantons in the $|P| \to \infty$ limit. If $\xi = 2$, they are of the same order, which still would not be desirable, unless perhaps δ can be chosen to be arbitrarily large.

This discussion motivates the following definition of the *core-dump exponent*, denoted ξ_{cd}:

Consider the set $\mathcal{S}$ of pairs $\{\Gamma(\beta_1, n_1), \Gamma(\beta_2, n_2)\}$ which admit flow trees making a nonzero contribution to $\Delta\Omega$. (Thus, the flow tree is based on core states stable at infinity, but unstable at $t^{1,2}_{\rm ms}$, or vice versa.) Let ξ_{cd} be the minimum of the set Ξ of numbers with the following property: For $\xi \in \Xi$, there exists a constant δ which is independent of $|P|$ such that, if we choose $\epsilon = \delta|P|^{-\xi}$ then $\mathcal{C}(P, \epsilon) \times \mathcal{C}(P; \epsilon)$ does not contain any of the states in $\mathcal{S}$.

From the argument given above, we know $\xi_{cd} \leq 3$. From the example (6.26) discussed above we also know that $\xi_{cd} \geq 1$. Then, as we have just explained, if $1 \leq \xi_{cd} \leq 2$ we will see below that a version of the (strong coupling) OSV conjecture can be proven. On the other hand, if it turns out that $2 < \xi_{cd}$ then the OSV conjecture (even at strong coupling) is almost certainly not correct. We can only say "almost certainly" because we have not excluded the possibility (however unlikely) that when we account for all swing states of a fixed charge their contributions to $\Delta\Omega$ magically sum to zero.

[50]catastrophic, that is, for the OSV conjecture

An independent argument sheds more light on why we must take $\xi_{cd} \geq 1$. We consider, a family of (β_P, n_P) (we will henceforth drop the subscript) such that for each P we have $P \cdot \beta = c_1|P|^2, n = c_2|P|^2$, where c_1, c_2 are constants. It turns out that such families exist for which $N_{DT}(\beta, -n) \neq 0$ [34]. On the other hand, we can attempt to build a boundstate using $\Gamma_1 = e^{P/2}\Gamma(\beta, n)$ and $\Gamma_2 = -\Gamma_1^*$. One finds that such a boundstate would have

$$\eta = \frac{24}{P^3}(P \cdot \beta - 2n) \tag{6.89}$$

and hence, the absence of superpolar states implies $\eta \geq 0$. On the other hand, the explicit examples of such families in [34] have constants c_1, c_2 such that $\eta < 0$. We are thus forced to conclude that states with such charges $\Gamma(\beta, n)$ are unstable at $t^1_{\rm ms}$, and hence such states provide examples of swing states. Note that, in this case, once again we can take $\epsilon = \delta/|P|$ for sufficiently small δ to eliminate such states from the ensemble $\mathcal{C}(P, \epsilon)$ thus proving once more that $\xi_{cd} \geq 1$.

In fact, a generalization of the argument of the previous paragraph offers a hint that in fact $\xi_{cd} = 1$. Suppose we have a family of (β_P, n_P) (we henceforth drop the subscripts) such that $\beta \cdot P = c_1|P|^\gamma$ and $n = c_2|P|^{\gamma'}$ and $N_{DT}(\beta, -n) \neq 0$. Reasoning as above, if $\gamma' > \gamma$, or $\gamma' = \gamma$ and $2c_2 > c_1$ then $\Gamma(\beta, n)$ is a swing state. On the other hand, we can (following [34]) use Castelnuovo's inequality (see [104], p. 252), which states that a curve of degree d in $\mathbb{C}P^n$ has its genus bounded above by $g < \frac{d^2}{2(n-1)}$ for $d \to \infty$. Next recall from the last equation in (4.23) that $n \leq g(\beta) - 1$. Thus, we should have $\gamma' \leq 2(\gamma - 1)$. Combining with $\gamma' \geq \gamma$ we see that $\gamma \geq 2$ for any such families. But then such swing states can always be eliminated with $\xi = 1$.

In section 7.5 we will give some more circumstantial evidence for $\xi_{cd} = 1$.

6.3.3 Factorization of the dilute gas D6-anti-D6 partition function

Let us now return to the analysis of (6.64). We assume that we define this partition function with $\epsilon = \delta|P|^{-\xi}$ with a suitable δ and ξ so that we can identify (6.74), (6.76) with (6.75),(6.77), respectively. In this case, we can proceed with the derivation of (6.94) below as follows:

We introduce the ϵ-dependent cut off version of (6.16):

$$\mathcal{Z}^\epsilon_{\rm D6-D2-D0}(u, v; B + iJ) := \sum_{(\beta,n)\in\mathcal{C}(P,\epsilon)} \Omega(\Gamma(\beta, n))|_{B+iJ}\ u^n\, v^\beta. \tag{6.90}$$

and using (6.73) we write:

$$\begin{aligned}\mathcal{Z}^\epsilon_{D6-\overline{D6}}(u, v, w) = u^{\frac{P^3+c_2P}{24}}\, w^{I_P} \sum_S u^{\frac{1}{2}PS^2} v^{PS} \\ \times \lim_{z\to\mathcal{L}^-} \mathcal{Z}^\epsilon_{D6-D2-D0}(w\, u, w^{-P}\, u^{-S-\frac{P}{2}}\, v^{-1}; zP) \\ \times \lim_{z\to\mathcal{R}^+} \mathcal{Z}^\epsilon_{D6-D2-D0}(w^{-1}u^{-1}, w^{-P}\, u^{S-\frac{P}{2}}\, v; zP).\end{aligned} \tag{6.91}$$

Please notice carefully the nature of the limits. Recall that, as we mentioned above, there is a subtlety when the halo particle has zero $D0$ charge, because in this case $t_{\rm ms}$ is exactly on the wall of marginal stability for pure D0-halos. As discussed there, this ambiguity can be resolved by perturbing the background B-field slightly, in which case there is a D0-halo contribution *either* on the first cluster *or* on the second. Therefore, as noted in eqs. (6.21-6.24), the case in which we first take $t \to \mathcal{L}^-$ does have the D0-halo MacMahon factor, while the case $t \to \mathcal{R}^+$ does not. Depending on the sign of the perturbation of the background B-field we have the limits $(\mathcal{L}^-, \mathcal{R}^+)$ as above or $(\mathcal{L}^+, \mathcal{R}^-)$. Our final answer will not depend on this dichotomy.

Now the identifications (6.21-6.24) imply

$$\lim_{z \to \mathcal{L}^-} \mathcal{Z}^{\epsilon}_{D6-D2-D0}(u, v; zP) = \mathcal{Z}^{\epsilon}_{DT}(u^{-1}, v) \tag{6.92}$$

$$\lim_{z \to \mathcal{R}^+} \mathcal{Z}^{\epsilon}_{D6-D2-D0}(u, v; zP) = \mathcal{Z}'^{\epsilon}_{DT}(u, v), \tag{6.93}$$

where the ϵ-dependent $\mathcal{Z}_{DT}$ is defined as the DT partition function with sum restricted to $(\beta, n) \in \mathcal{C}(P, \epsilon)$. Similarly, for $\mathcal{Z}'^{\epsilon}_{DT}$, we take the infinite product $\mathcal{Z}'_{DT}$ and truncate its series expansion. In terms of these quantities we can therefore write:

$$\begin{aligned}\mathcal{Z}^{\epsilon}_{D6-\overline{D6}}(u, v, w) = u^{\frac{P^3+c_2 P}{24}} w^{I_P} \sum_S u^{\frac{1}{2}PS^2} v^{PS} \\ \times \mathcal{Z}^{\epsilon}_{DT}(w^{-1}u^{-1}, w^{-P}\, u^{-S-\frac{P}{2}}\, v^{-1}) \\ \times \mathcal{Z}'^{\epsilon}_{DT}(w^{-1}u^{-1}, w^{-P}\, u^{S-\frac{P}{2}}\, v).\end{aligned} \tag{6.94}$$

Equation (6.94) is the main result of this section. As noted above, the sum over S is a formal sum, but the coefficients of $u^n v^\beta w^\ell$ are well-defined.

6.4 D4-D2-D0 degeneracies

In this section we finally return to D4-D2-D0 degeneracies and use the technology developed above to present a derivation of our refined OSV formula eq. (6.113). We first relate the polar part of the D4-D2-D0 partition function to the D6-antiD6 dilute gas partition function. This introduces an error, but one well-controlled by the extreme polar state conjecture (provided it turns out that $\xi_{cd} \leq 2$). We then combine this with the fareytail expansion. At the end of the section we discuss the error terms in the refined OSV formula.

6.4.1 Approximate factorization of polar D4 partition function

Now we return to the considerations of section 2. The extreme polar state part of the D4 partition function (2.39) is, using the notations of section 2.4

$$\mathcal{Z}^{\eta_*}(\tau, \bar\tau, C) := \sum_\gamma H^{\eta_*}_\gamma(\tau) \Psi_\gamma(\tau, \bar\tau, C) \tag{6.95}$$

$$H^{\eta_*}_\gamma(\tau) := \sum_{\eta < \eta_*} \Omega([\gamma, \tfrac{P^3}{24} - \eta \tfrac{P^3}{24}], t = i\infty)\, e^{-2\pi i \tau \frac{P^3}{24} + 2\pi i \tau \eta \frac{P^3}{24}} \tag{6.96}$$

where $\eta_* \ll 1$ and $\Omega([\gamma, \hat{q}_0])$ is as defined above (2.46). When $\mathrm{Im}\,\tau$ is sufficiently large,[51] the extreme polar part of the partition function is a good approximation to the full polar part $\mathcal{Z}^-$ (which is obtained by taking $\eta_* = 1$):

$$H_\gamma^-(\tau) = H_\gamma^{\eta_*}(\tau) \times \Big(1 + \mathcal{O}(e^{-\Delta(P,\eta_*,\tau)P^3})\Big) \tag{6.97}$$

where

$$\Delta(P, \eta_*, \tau) := \min_{1>\eta>\eta_*} \Big(-\Sigma(P,\eta) + \frac{\pi}{12}\mathrm{Im}\,\tau\,\eta\Big) \tag{6.98}$$

$$\Sigma(P,\eta) := \frac{1}{P^3} \max_\gamma \log \Big|\Omega([\gamma, \tfrac{P^3}{24} - \eta\tfrac{P^3}{24}])\Big| \,. \tag{6.99}$$

Note that as long as $\Delta(P, \eta_*, \tau)$ is positive and doesn't decay as $|P|^{-3}$ or faster, the error is exponentially small when P^3 is large; in particular this is the case for $\mathrm{Im}\,\tau$ sufficiently large.

To get an idea of the general behavior of the error term, and to justify our notations separating out the P^3 factor, let us assume for the moment that we can estimate the growth of the right hand side of (6.99) from the growth of the entropy of two-centered configurations of the kind analyzed in section 3.4, and more specifically for further simplicity let us restrict to configurations with $\tilde{n} = 0$ in the notation used there. Then $\tilde{\beta} = \eta/24$ and the total Bekenstein-Hawking entropy of the two centers is $S \sim \tilde{\beta}^{3/2}P^3 \sim \eta^{3/2}P^3$. Hence this estimates $\Sigma(P,\eta) \approx c\eta^{3/2}$ for some constant c independent of P. In this case we have $\Delta(P, \eta_*, \tau) > 0$ if and only if $\mathrm{Im}\,\tau > 12c/\pi$. If $\mathrm{Im}\,\tau$ drops below this critical value, our error estimate blows up. Moreover, η_* should not become too small if we want the error to be exponentially small, since Δ is in any case smaller than $\frac{\pi}{12}\eta_*\mathrm{Im}\tau - c\eta_*^{3/2}$.

Now the index Ω in (6.99) actually receives contributions from many other flow trees, some very complicated, so this simple estimate might be too naive. However, it will at least give a rough lower bound on the actual growth of Σ, unless miraculous almost-exact cancelations occur between contributions of different signs to the index. This implies in particular that our approximations break down for $\mathrm{Im}\,\tau$ less than some order 1 critical value, unless these miraculous cancelations occur. We will discuss this potential breakdown in detail in section 7, and show that it is not due to a failure of our derivation, but intimately related to the entropy enigma of section 3.5.

Combining (6.97) and (6.95), we can write

$$\mathcal{Z}^-(\tau, \bar{\tau}, C) = \mathcal{Z}^{\eta_*}(\tau, \bar{\tau}, C) \times \Big(1 + \mathcal{O}(e^{-\Delta(P,\eta_*,\tau)P^3})\Big) \,. \tag{6.100}$$

If we make the OSV substitution (2.17) and formally put $\bar{\tau} = \tau$, then both sides of this equation diverge due to the non-definiteness of the intersection product on L_X, so the error estimate, strictly speaking, is not meaningful. However we can still give it a precise

[51]Further on we will apply the approximate factorization we are currently deriving in the fareytail expansion (2.71), and will find that the dominant term for our purposes comes from the term corresponding to $A = S$, where τ gets replaced by $-1/\tau$. Thus, in these applications, we will need a sufficiently large $\mathrm{Im}\,(-1/\tau)$ to get approximate factorization.

meaning by considering the terms for a fixed D2-charge (i.e. a fixed power of $e^{-2\pi i C}$), or equivalently by multiplying both sides by some $e^{2\pi i C\cdot Q}$ and integrating out C. (After a modular transformation the integral on C involves one wrong sign Gaussian integral which is easily evaluated in the usual analytically continued sense). Alternatively, we can just cut off the sum over L_X in the theta-functions. Since the error estimate provides a *relative* error, it has a well defined meaning for any such finite truncation. In the end we can take the cutoff to infinity, which when computing any physically meaningful quantity should give a finite result, with a well-defined error estimate. Keeping this interpretation of the error term in mind, we will from now on put $\bar{\tau} = \tau$.

The extreme polar state conjecture implies that there exists an $\epsilon(\eta_*) \sim \eta_*$ such that all extreme polar states (with $\eta < \eta_*$) are generated by D6-anti-D6 dilute gas pairs with $(\beta_i, n_i) \in \mathcal{C}(P, \epsilon)$. We now invoke the formula (5.4) for the polar degeneracies and recall eq. (6.67). Combining this with (6.100) at $\bar{\tau} = \tau$ then gives

$$\begin{aligned} \mathcal{Z}^-(\tau, C) \,=\, & \frac{1}{2\pi}\frac{\partial}{\partial\alpha}\mathcal{Z}^{\epsilon}_{\mathrm{D6}-\overline{\mathrm{D6}}}(e^{-2\pi i\tau}, e^{-2\pi i(C+\frac{P}{2})}, e^{2\pi(\alpha-\frac{i}{2})})\Big|_{\alpha=0} \\ & \times\Big(1+\mathcal{O}(e^{-\Delta(P,\eta_*,\tau)P^3})\Big) \end{aligned} \tag{6.101}$$

where $\mathcal{Z}^{\epsilon}_{\mathrm{D6}-\overline{\mathrm{D6}}}$ was defined in (6.64). Note that although some D6-anti-D6 pairs with $(\beta_i, n_i) \in \mathcal{C}(P, \epsilon)$ will have $\eta > \eta_*$, they will nevertheless still all be polar states (assuming ϵ sufficiently small), and therefore not affect the error any further.

Now from (6.94), we get

$$\begin{aligned} & \mathcal{Z}^{\epsilon}_{D6-\overline{D6}}(e^{-2\pi i\tau}, e^{-2\pi i(C+\frac{P}{2})}, e^{2\pi(\alpha-\frac{i}{2})}) \\ & = e^{-2\pi i\tau\frac{P^3+c_2P}{24}}\, e^{2\pi I_P(\alpha-\frac{i}{2})}\sum_S e^{-\pi i\tau PS^2-2\pi i(C+\frac{P}{2})PS} \\ & \quad\times \mathcal{Z}^{\epsilon}_{DT}(-e^{2\pi i(\tau+i\alpha)}, e^{2\pi i[i\alpha P+\tau(S+\frac{P}{2})+C]}) \\ & \quad\times \mathcal{Z}'^{\epsilon}_{DT}(-e^{2\pi i(\tau+i\alpha)}, e^{2\pi i[i\alpha P+\tau(-S+\frac{P}{2})-C]}). \end{aligned} \tag{6.102}$$

Let us make two remarks:

1. To get the factorized generating function (6.94), we needed to restrict to $(\beta_i, n_i) \in \mathcal{C}(P, \epsilon)$, as defined in (6.51), and take $\epsilon = \delta|P|^{-\xi_{cd}}$ to dump swing states. According to the extreme polar state conjecture, we should therefore take $\eta_* = \frac{\delta}{\mu}|P|^{-\xi_{cd}}$. The cutoff restricts the sum to states splitting into a rank $r = 1$ D6 and anti-D6; in the picture of [17, 18], this corresponds to leaving out $\mathbb{Z}_r$-quotients of $\mathrm{AdS}_3 \times S^2$ with $r > 1$. Furthermore, dumping swing states such as our example in section 6.3.2 corresponds in M-theory to dumping certain black M5 rings which exist in infinite radius Taub-NUT but not on the finite size S^2. See section 7.3 below for a more extensive discussion.

2. Again, due to the divergence of the sum over S, the error estimate in (6.101) is strictly speaking meaningless, but as in the discussion under (6.100) we can give it a precise meaning e.g. by introducing a cutoff in the sum over S. Finally, putting (6.101) and (6.102) together thus gives an approximate factorization formula for $\mathcal{Z}^-(\tau, C)$.

6.4.2 Derivation of OSV

We are finally ready to put all our results together and derive a refined version of the OSV conjecture. From (2.18), we have

$$\Omega((0,P,Q,q_0);t=i\infty) = (-i)^{h+1}\oint d\phi^0 d\Phi\, e^{-2\pi q_\Lambda \phi^\Lambda}\mathcal{Z}(\tau=\bar\tau=i\phi^0, C=i\Phi-\frac{P}{2}) \quad (6.103)$$

where the ϕ^Λ-integrals run over a single imaginary period on the imaginary axis. We wish to derive an OSV-like formula in the case $\Gamma=(0,P,Q,q_0)$ is *nonpolar* and large, and hence has a single centered black hole realization. The first step is to substitute the fareytail expansion (2.71) of $\mathcal{Z}$ into (6.103). Since we are considering nonpolar terms, the $c=0$ part of the series will not contribute to (6.103). To leading order in the saddle point approximation, the $c\neq 0$ terms contribute terms of order $e^{S_{\rm sugra}/c}$, as can be seen directly from the expressions (or see e.g. appendix A of [11]). Hence in the large charge limit, the $c>1$ terms will be suppressed by a factor $\sim e^{-kS_{\rm sugra}}$ compared to the $c=1$ terms, with k some order 1 constant. Alternatively, we can say that these terms are effectively suppressed by a factor $\sim e^{-kP^3/\phi^0}$ in the partition function $\mathcal{Z}(P,\phi)$, with k some constant of order one, and where we take ϕ^0 to be positive (as it is at the saddle point).

The $c=1$ terms correspond to $SL(2,\mathbb{Z})$ elements

$$A=\begin{pmatrix}0 & -1\\ 1 & d\end{pmatrix} = S\,T^d \quad (6.104)$$

so we can write, with $\tau=\bar\tau=i\phi^0$, $C=i\Phi-\frac{P}{2}$,

$$\begin{aligned}\mathcal{Z}^+(\tau,C) &= \sum_{d\in\mathbb{Z}}\omega_S^{-1}\omega_T^{-d}(\tau+d)\,e^{2\pi i\frac{C^2}{2(\tau+d)}}\,\mathcal{Z}^-(-\frac{1}{\tau+d},\frac{C}{\tau+d}) && (6.105)\\ &\quad\times\Big(1+\mathcal{O}(e^{-kP^3/(\tau+d)})\Big)\,. && (6.106)\end{aligned}$$

Here $\mathcal{Z}^+$ is the non-polar part of $\mathcal{Z}$, and the error term comes from dropping the $c>1$ terms as discussed above. Note that although the error is no longer exponentially small when $d\to\infty$, the large d terms themselves are exponentially suppressed compared to the small d terms, and therefore unimportant. Put differently, the sum over d can be traded for extending the integration contour in (6.103) over the entire imaginary ϕ^0-axis, but the large d terms will correspond to points far away from the saddle point, and are therefore unimportant.

From (6.105), (6.101) and (6.102), we find, after some work and substituting $\tau=\bar\tau=i\phi^0$, $C=i\Phi-\frac{P}{2}$:

$$\begin{aligned}\mathcal{Z}^+_{\rm BH}(\phi^0,\Phi) = \frac{1}{2\pi}\frac{\partial}{\partial\alpha}\Big|_{\alpha=0}&\Bigg(\sum_{d,\hat S} i(\phi^0-id)e^{2\pi I_P\alpha}e^{\frac{2\pi}{\phi^0-id}\frac{P^3+c_2P}{24}-\frac{\pi}{\phi^0-id}(\Phi+i\hat S)^2-2\pi i\frac{P}{2}\cdot\hat S-2\pi i\frac{c_2P}{24}d}\\ &\times\mathcal{Z}^\epsilon_{DT}(-e^{-2\pi(\frac{1}{\phi^0-id}+\alpha)},e^{2\pi i(\frac{1}{\phi^0-id}(\Phi+i\hat S+i\frac{P}{2})+i\alpha P)})\\ &\times\mathcal{Z}'^\epsilon_{DT}(-e^{-2\pi(\frac{1}{\phi^0-id}+\alpha)},e^{2\pi i(\frac{1}{\phi^0-id}(-\Phi-i\hat S+i\frac{P}{2})+i\alpha P)})\Bigg)\\ &\times\Bigg(1+\mathcal{O}(e^{-\Delta(P,\eta_*,\frac{i}{\phi^0-id})\,P^3})\Bigg) \end{aligned}\quad (6.107)$$

Here $\hat{S} := S + \frac{P}{2} \in H^2(X, \mathbb{Z})$, and various complicated phase factors have canceled in a nontrivial way.

Finally, using the identification $\mathcal{Z}'_{DT}(-e^{-g}, e^{2\pi i t}) = \mathcal{Z}'_{GW}(g, t)$ discussed in section 1.3, this becomes, remarkably

$$\mathcal{Z}^+_{\rm BH}(\phi^0, \Phi) = \frac{1}{2\pi} \frac{\partial}{\partial \alpha}\Big|_{\alpha=0} \sum_{d,\hat{S}} i(\phi^0 - id)\, e^{\mathcal{F}^\epsilon(P,\phi^0-id,\Phi+i\hat{S},\alpha)-2\pi i\frac{P}{2}\cdot\hat{S}-2\pi i\frac{c_2 P}{24}d} e^{\delta\mathcal{F}} \tag{6.108}$$

where

$$\mathcal{F}^\epsilon(P, \phi, \alpha) := F^\epsilon_{\rm top}(g, t) + \overline{F^\epsilon_{\rm top}(g, t)}, \tag{6.109}$$

$$F^\epsilon_{\rm top} := \log \mathcal{Z}^\epsilon_{\rm top} = \log \mathcal{Z}_{\rm pol} + \frac{1}{2}(\log \mathcal{Z}^\epsilon_{\rm DT} + \log \mathcal{Z}'^{,\epsilon}_{\rm DT}) \tag{6.110}$$

with substitutions

$$g \equiv \frac{2\pi}{\phi^0} + 2\pi\alpha, \qquad t \equiv \frac{1}{\phi^0}(\Phi + i\frac{P}{2}) + i\alpha P \tag{6.111}$$

and error

$$\delta\mathcal{F} = \mathcal{O}(e^{-\Delta(P,\eta_*,\frac{i}{\phi^0})\,P^3}). \tag{6.112}$$

Recall that the cutoff ϵ is related to η_* through the extreme polar state conjecture of section 6.2.2 as $\epsilon = \mu\eta_*$, and that we took $\epsilon = \delta|P|^{-\xi_{cd}}$ to get rid of swing states. In taking the complex conjugate in (6.109), ϕ^0, Φ and α should formally be taken real. We also dropped terms of quadratic and higher order in α, since we set $\alpha = 0$ after taking the derivative. Surprisingly perhaps, the $2\pi I_P \alpha$ in the exponent of (6.107) is reproduced to this order by the α-dependence of $F_{\rm pol}$ after substituting (6.111). The inclusion of ϕ^0 in the measure and the ∂_α operation were absent in the original OSV conjecture [10]. They can be traced back respectively to the fact that $\mathcal{Z}(\tau, C)$ has modular weight $(-3/2, 1/2)$ (for a proper $SU(3)$ holonomy Calabi-Yau), and to the fact that there is a factor $\sim |\langle\Gamma_1, \Gamma_2\rangle|$ in (5.4). Both modifications are in agreement with the results of section 2.3 (see eq. (2.54)) obtained in the small ϕ^0 limit by arguments independent of our D6-anti-D6 picture.

Note furthermore that the sum over $\hat{S}$ and d together with the phase factors in (6.108) give the right hand side precisely the same periodicity as the left hand side. These terms also allow us to invert (6.108) to the simple expression

$$\Omega(0, P, Q, q_0; t = i\infty) = \int d\phi\, \mu(P, \phi)\, e^{-2\pi q_\Lambda \phi^\Lambda}\, e^{\mathcal{F}^\epsilon(P,\phi)+\delta\mathcal{F}}. \tag{6.113}$$

where the "measure factor" $\mu(P, \phi)$ is

$$\mu(P, \phi) = \frac{(-i)^h}{2\pi}\, \phi^0\, \frac{\partial}{\partial\alpha} \mathcal{F}^\epsilon(P, \phi, \alpha)\,|_{\alpha=0} = (-i)^h \phi^0 I_P + \text{inst. corr.} \tag{6.114}$$

Note that these instanton corrections are of order $\exp[-2\pi|P|/\phi^0]$ which is the same order as the other terms we are trying to keep track of and hence are rather essential to a correct formulation of the OSV conjecture.

Rather curiously, the measure factor can also be written as

$$\mu(P,\phi) = (-i)^h \frac{4\pi}{g^2}\, e^{-K^\epsilon} \tag{6.115}$$

where K^ϵ is a generalized Kähler potential, defined by

$$e^{-K^\epsilon} = \mathrm{Re}\Big[\bar{X}^{\Lambda_1} I_{\Lambda_1}{}^{\Lambda_2} \frac{\partial F^\epsilon_{\text{top}}}{\partial X^{\Lambda_2}}\Big] \tag{6.116}$$

with

$$X^0 = 2i\phi^0 \qquad\qquad X^A = (\Phi + \frac{i}{2}P)^A \tag{6.117}$$

and we have used the property that ϕ^0, Φ and P are real. Recall that $I_{\Lambda_1}^{\Lambda_2}$ was defined to be $I_{\Lambda_1}^{\Lambda_2} = \sigma_{\Lambda_2}\,\delta^{\Lambda_1}_{\Lambda_2}$ where $\sigma_0 = 1$, $\sigma_A = -1$. This measure factor is the same as the one found in [13] for $X = T^6$ and $X = T^2 \times K3$.[52]

6.4.3 Analysis of the error terms

Let us now consider the error term. When the saddle point lies at sufficiently small ϕ^0, Δ is guaranteed to be positive, hence the first error term is exponentially small at large P. As we mentioned before, the meaning of "sufficiently small" depends on the growth of the polar entropies $S(\eta) \sim \Sigma(P,\eta)P^3$ as a function of P. We define an exponent κ by $\Sigma(P,\eta)P^3 \sim |P|^\kappa$, or, more precisely [53]

$$\kappa := 3 + \overline{\lim}_{|P|\to\infty} \frac{\log \Sigma(P,\eta)}{\log |P|}. \tag{6.118}$$

Then we need

$$g^{-1} \sim \phi^0 < \mathcal{O}(|P|^{3-\kappa}) \tag{6.119}$$

for all η, $0 \le \eta \le 1$, in order to have an exponentially suppressed error. A simple estimate based on the BH entropy of two-centered configurations realizations of polar states, outlined under (6.99), indicates $\kappa = 3$, hence we would need $g > \mathcal{O}(1)$, i.e. strong topological string coupling. However, since we are considering indices it might be possible in principle that miraculous cancelations occur which effectively lower κ. We postpone further discussion of this possibility to section 7.4.2.

Let us now consider the error we have when we are well within the regime (6.119). More precisely let ϕ^0_{cr} be the value at which $\Delta = 0$. For $\phi^0 \ll \phi^0_{cr}$, the second term dominates in (6.98), so

$$\Delta \cong \frac{\pi}{12}\frac{1}{\phi^0}\eta_* \tag{6.120}$$

and the error term becomes

$$\delta\mathcal{F} \sim \mathcal{O}(e^{-\frac{\pi}{12}\frac{\eta_* P^3}{\phi^0}}) \tag{6.121}$$

[52] There are slight differences coming from the different power of ϕ^0 in (6.114), namely $(\phi^0)^{1-b_1}$, one gets in those cases, and the possibility to include gravitini charges. These factors arise in our approach for these cases as well.

[53] We take the limit supremum $\overline{\lim}$ which always exists. We expect κ to be at most weakly dependent on η.

For η_* a constant independent of P, this agrees with the error found in [13] for $T^2 \times K3$ and T^6. However, as discussed in section 6.3.2, in general we must take ϵ, and hence $\eta_* = \epsilon/\mu$, to depend on $|P|$. Taking $\epsilon = \delta|P|^{-\xi_{cd}}$ these lead to corrections of order

$$\exp[-\frac{\pi\delta}{12\mu}\frac{P^{3-\xi_{cd}}}{\phi^0}] \tag{6.122}$$

We saw that the split configurations (6.26) imply that we must have $\xi_{cd} \geq 1$. On the other hand, worldsheet instantons contribute terms of order $\exp[-2\pi\beta \cdot P/\phi^0]$. Therefore if the corrections (6.122) are not to overwhelm the worldsheet instanton corrections then we must have $\xi_{cd} \leq 2$. Unfortunately the value of ξ_{cd} is unknown. We hasten to point out that the value $\xi_{cd} = 1$ makes excellent physical sense for reasons discussed in section 7.5 below.

7. Discussion

7.1 Summary

Let us summarize our final result, and discuss to what extent it agrees with the original OSV conjecture.

We consider the index of BPS states, defined in (1.6), with charge $p^0 = 0$, large $|P| := (D_{ABC}P^AP^BP^C)^{1/2}$ and $\widehat{q}_0 := q_0 - \frac{1}{2}D^{AB}Q_AQ_B < 0$. For these charges a single centered black hole solutions exists. We choose the background $t = i\infty$. Then the index is given by

$$\Omega(P, Q, q_0; t = i\infty) = \int d\phi\, \mu(P, \phi)\, e^{-2\pi q_\Lambda \phi^\Lambda}\, e^{\mathcal{F}^\epsilon(P,\phi)+\delta\mathcal{F}}. \tag{7.1}$$

Where, using the substitutions

$$g \equiv \frac{2\pi}{\phi^0}, \qquad t^A \equiv \frac{1}{\phi^0}(\phi^A + i\frac{P^A}{2}), \tag{7.2}$$

we have

$$\begin{aligned}
\mu(P,\phi) &:= (-i)^h \frac{4\pi}{g^2}\, e^{-K^\epsilon(g,t,\bar{t})} = (-i)^h \phi^0 (\tfrac{P^3}{6} + \tfrac{c_2 P}{12}) + \text{inst. corr.} && (7.3)\\
\mathcal{F}^\epsilon(P,\phi) &:= F^\epsilon_{\text{top}} + \overline{F^\epsilon_{\text{top}}}, && (7.4)\\
F^\epsilon_{\text{top}}(g,t) &:= \log \mathcal{Z}^\epsilon_{\text{top}}(g,t) && (7.5)\\
&:= \log \mathcal{Z}_{\text{pol}}(g,t) + \frac{1}{2}\Big(\log \mathcal{Z}^\epsilon_{\text{DT}}(-e^{-g}, e^{2\pi i t}) + \log \mathcal{Z}'^{,\epsilon}_{\text{DT}}(-e^{-g}, e^{2\pi i t})\Big) && (7.6)\\
\mathcal{Z}_{\text{pol}}(g,t) &:= \exp\Big(-\frac{(2\pi i)^3}{6g^2} D_{ABC} t^A t^B t^C - \frac{2\pi i}{24} c_{2A} t^A\Big) && (7.7)\\
\mathcal{Z}^\epsilon_{DT}(u,v) &:= \sum_{|n|,\beta\cdot P < \epsilon P^3} N_{DT}(\beta, n)\, u^n\, v^\beta. && (7.8)
\end{aligned}$$

and $\mathcal{Z}'^{,\epsilon}_{\text{DT}}$ is defined analogously by cutting off the series for $\mathcal{Z}'_{\text{DT}}$. The full expression for e^{-K^ϵ} is given in (6.116). In taking the complex conjugate in (7.4), ϕ should be treated as real.

The error is of order

$$\delta\mathcal{F} = \mathcal{O}(e^{-\Delta(P,\eta_*,\frac{i}{\phi^0})\,P^3}) \tag{7.9}$$

where

$$\Delta(P,\eta_*,\frac{i}{\phi^0}) := \min_{\eta_*<\eta<1}\left(-\Sigma(P,\eta)+\frac{\pi}{12\,\phi^0}\,\eta\right) \tag{7.10}$$

$$\Sigma(P,\eta) := \frac{1}{P^3}\max_{[Q]}\log\left|\Omega\left(P,[Q],\hat{q}_0=(1-\eta)\tfrac{P^3+c_2P}{24}\right)\right|. \tag{7.11}$$

Here $\Omega\,(P,[Q],\hat{q}_0)$ is the index of BPS states of D4-charge P, D2-charge $[Q]$ with $[Q]$ in a fundamental domain for the symmetry $Q_A \to Q_A + D_{ABC}P^Bn^C$, $n^C \in \mathbb{Z}$, and reduced D0-charge (invariant under this symmetry) $\hat{q}_0$, evaluated in a background $t = i\infty$. The range $0 \leq \eta < 1$ corresponds to polar charges, with $\eta = 0$ being the most polar one.

The integral in (7.1) runs over the imaginary ϕ-axes. One might worry that the error (6.112) becomes $\mathcal{O}(1)$ for large ϕ^0, but since the integral is dominated by its saddle point, this part of the integration contour is negligible anyway. The saddle points for ϕ turn out to be real.

For the cutoffs ϵ and η, we have the relations

$$\mu\,\eta_* = \epsilon = \delta|P|^{-\xi_{cd}}. \tag{7.12}$$

The first equality follows from the extreme polar state conjecture of section 6.2.2, and the second one is required to get rid of swing states, as discussed in section 6.3.2. Here δ and μ are P-independent constants. The core dump exponent ξ_{cd} is a kind of critical exponent, which we bounded by $1 \leq \xi_{cd} \leq 3$.

In order to clarify the domain of validity of (7.1) we introduced a second critical exponent, defined by the growth of the polar state indices growth with P at fixed η:

$$\log|\Omega| \sim |P|^\kappa \tag{7.13}$$

or, more precisely, as in (6.118). In terms of κ the error term is controlled only if

$$g > \mathcal{O}(|P|^{\kappa-3}) \tag{7.14}$$

If we estimate the polar index growth by the growth of the two-centered black hole realizations, we get

$$\kappa = 3 \tag{7.15}$$

implying OSV is only valid at *strong* topological string coupling. We discuss the possibility that κ gets miraculously lowered by cancelations in the index in section 7.4. Neglecting instanton corrections, the saddle point value of ϕ^0 is given by $\phi^0_* \approx \sqrt{\frac{P^3}{24|\widehat{q}_0|}}$. Hence the bound (7.14) translates to a bound on the charges:

$$|\widehat{q}_0| > \mathcal{O}(|P|^{2\kappa-3}). \tag{7.16}$$

Note that when $\kappa = 3$, this bound always gets violated when scaling up the total charge $\Gamma = (0, P, Q, q_0)$ uniformly by a sufficiently large Λ, and that to avoid this, we need $\kappa \leq 2$. This is not an artifact of our derivation scheme: it is precisely as expected from the "entropy enigma" discussed in section 3.5. The closely related two centered configurations considered there had an entropy growing as Λ^3, so if no miraculous cancelations occur between contributions to the index bringing down the growth of $\log \Omega(\Lambda\Gamma)$ to Λ^2 or lower, the OSV conjecture at $t = i\infty$ necessarily breaks down at sufficiently large Λ, since it predicts $\Omega(\Lambda\Gamma) \sim \Lambda^2$ to leading order at large Λ. We will discuss this in more detail in section 7.4.

Finally, when g is well inside the regime (7.14), the error simplifies to

$$\delta\mathcal{F} \sim \exp[-\frac{\pi\delta}{12\mu}\frac{P^{3-\xi_{cd}}}{\phi^0}] \tag{7.17}$$

For this to be negligible compared to the instanton contributions to the free energy, which are suppressed as $e^{-\beta\cdot P/\phi^0}$, we need $\xi_{cd} < 2$ or $\xi_{cd} = 2$ and $\delta \gg 1$. Using $\mathrm{Im}t \sim P/\phi^0$ and $g \sim 1/\phi^0$, we can also write this as

$$\delta\mathcal{F} \sim \exp[-\frac{c}{g^{2-\xi_{cd}}}|\mathrm{Im}t|^{3-\xi_{cd}}], \tag{7.18}$$

with c a constant. Note that for $\xi_{cd} = 1$, this is suggestive of a D4/M5 contribution to the Schwinger computation of the topological string free energy [32, 33]. We return to the issue of determining the value of ξ_{cd} in section 7.5.

7.2 Differences with original OSV conjecture

We note the following differences with the original OSV conjecture:

1. There is an additional measure factor $\mu(P, \phi)$, in agreement with the special cases studied in [11, 12, 13]. This does not affect the leading saddle point evaluation of the entropy, but does affect the inverse charge corrections to it. The origin of this measure factor is essentially the presence of the angular momentum factor $|\langle\Gamma_1, \Gamma_2\rangle|$ in the factorization formula (5.4) for degeneracies of polar states.

   Furthermore, if we state the OSV formula using $|\Psi_{\rm top}|^2$ with the standard definition of $\Psi_{\rm top}$ then, since our definition of the topological string partition function in (6.110) was nonstandard (because the degree zero terms $\mathcal{Z}^0_{\rm DT}$ make use of the MacMahon function, which differs from the standard perturbative $F_{\rm top}$ by a term proportional to $\frac{\chi}{24}\log g$) one would have to include a further factor of $g^{\chi/24}$ in the measure.

2. The topological string partition function is cut off. The cutoff cannot be removed, since the full $\mathcal{Z}_{\rm top}$ has zero radius of convergence and hence does not exist as a function which can be integrated, not even in a saddle point approximation. We were led to put a cutoff on the DT invariants $N_{DT}(\beta, n)$ contributing to $\mathcal{Z}_{\rm top}$, namely $|n| < \epsilon P^3$, $\beta \cdot P < \epsilon P^3$. This does not translate into a simple cutoff on the corresponding M2 BPS invariants in the infinite product representation. Physically, this happens because the existence of D6-anti-D6 bound states depends on the *total* D2-D0 charge

of the constituents, while the M2 BPS invariants refer to "one particle" contributions to these D2-D0 charges. As usual with cutoffs, there is some arbitrariness in their choice; possibly there are other "regularization schemes" than the one we used.

3. We find corrections, exponentially suppressed at large P. These are due to the $c > 1$ terms of the fareytail series as well as to the non-extreme polar states we dropped. Taking into account terms corresponding to $SL(2,\mathbb{Z})$ transforms with $c > 1$ in the fareytail series would add $|\mathcal{Z}_{\rm top}|^2$ type terms to the integrand on the right hand side of the OSV formula, but with substitutions different from (7.2). These give corrections $\delta\mathcal{F} \sim e^{-g\,P^3}$. Taking into account contributions of D6-anti-D6 bound states with D6 multiplicities $r > 1$ (which are necessarily non-extreme polar with $\eta \geq 3/4$) presumably would spoil the simple relation to $\mathcal{Z}_{\rm top}$. These contributions give corrections $\delta\mathcal{F} \sim e^{-g\,P^3}$. Moreover, taking into account non-extreme polar states ($\eta > \eta_*$), even at $r = 1$, would spoil factorization, as suggested e.g. already by fig. 17. More importantly, the existence of the swing states of section 6.3.2, which spoil factorization and the relation to DT invariants, force us to restrict to $\eta < \eta_* \sim |P|^{-\xi_{cd}}$. This gives corrections $\delta\mathcal{F} \sim e^{-g\,|P|^{3-\xi_{cd}}}$. If $\xi_{cd} > 2$, the error actually swamps the instanton contributions we want to keep. We only know with certainty that $1 \leq \xi_{cd} \leq 3$, although there is some evidence that $\xi_{cd} = 1$.

4. We find a restriction on the range of validity. We need (7.16) to be satisfied. In particular, if $\kappa > 2$, our formula breaks down when uniformly scaling up all charges by a sufficiently large Λ, whereas the original conjecture was meant to be valid precisely in this large Λ regime. Put differently, since the saddle point $g_* \sim 1/\Lambda$, our result is guaranteed to work in the strong topological coupling regime, but fails in the weak coupling regime unless there are miraculous cancelations between the contributions to the indices of the polar states. The original conjecture on the other hand was supposed to work at weak g.

Clearly, the last two points lead to potentially the most significant discrepancies with the original conjecture, so we will examine these points more closely and discuss the various possible loopholes in sections 7.4 and 7.5. Before we get to this though, we will give an interpretation of our results in the language of the M-theory derivations [17, 18] of the OSV conjecture which have appeared, and demonstrate in particular that the discrepancies and subtleties we find are not tied to our specific picture.

7.3 Comparison with M-theory derivations

The M-theory derivations [17, 18] of the OSV conjecture did not detect the measure factor and did not attempt to give bounds on the regime of validity or on the error. The appearance of a cutoff and corrections was emphasized in [18], but the level of analysis was insufficient to provide explicit cutoff prescriptions or estimates of corrections.

In these derivations the polar states were represented as dilute gasses of spinning M2 branes and anti-M2 branes orbiting the poles of the S^2 in the spacetime $AdS_3 \times S^2 \times X$, the latter carrying a G-flux proportional to P. Some of the issues which were left open in [17, 18] were the parameter range for which this dilute gas approximation is accurate, what happens if the M2's and anti-M2's start "spilling over" into each others hemispheres and what the effect is of other BPS states such as M5 branes and of taking into account other geometries such as quotients of $AdS_3 \times S^2$.

The various elements in our picture have a fairly straightforward translation into this M-theory picture, and hence our analysis clarifies all of these issues.[54] The rough idea is as follows. A D6-brane lifts in M-theory to a Taub-NUT space, so a bound state consisting of a pure D6 and an anti-D6 stabilized by flux lifts to a Taub-NUT-anti-Taub NUT geometry stabilized by flux. In the M-theory limit, this geometry becomes $AdS_3 \times S^2$ with flux proportional to the net D4-charge P, with the north and south poles of the sphere identified with the centers of Taub-NUT, projecting down to the D6 and resp. anti-D6. Adding D2D0 charge to the D6 and anti-D6 branes to turn them into black holes corresponds to putting spinning M2 BMPV black holes at the north and south poles of the sphere [70]. Our D2D0 halos orbiting the D6 and anti-D6 lift to M2 branes orbiting the north and south poles. More complicated D6 core states such as the swing states in section 6.3.2 realized as 2-centered D6-D4 configurations become M5 black rings in $AdS_3 \times S^2$. Higher rank $r > 1$ D6 anti-D6 bound states correspond to $\mathbb{Z}_r$ quotients of $AdS_3 \times S^2$ and deformations thereof. Fractal flow trees such as fig. 8 lift to foamy "bubbling" geometries. And so on. The upshot is that we have included *all* possible contributions in our analysis, no matter how exotic or complicated.

Now, the idea of [17] can be summarized like this: Cut the $AdS_3 \times S^2$ in two halves and identify one half with part of Taub-NUT with flux and the other one with part of anti-Taub-NUT with flux. Now complete these cut off, finite-size, finite flux Taub-NUT spaces to complete Taub-NUT spaces of infinite size, with infinite Taub-NUT circle radii and infinite total integrated flux, and count BPS states on each of those. In this infinite-size, infinite flux limit, all BPS states are well-described by lowest Landau-levels of spinning M2 probes, and the generating function of their indices is the Gopakumar-Vafa partition function (as we argued in section 6.1.2, refining the analysis of [29]). Hence, in this approximation, ignoring higher r geometries, finite size effects and the coupling between the two sectors, the $AdS_3 \times S^2$ elliptic genus is simply given by the product of two GV products, leading to $\mathcal{Z} \sim \mathcal{Z}_{\rm top}\overline{\mathcal{Z}_{\rm top}}$.

It is of course not clear to what extent this picture is justified, and through the above dictionary, our work can be interpreted as a thorough analysis of this problem. Let us translate a few of our results into this M-theory picture:

- We found it necessary in section 6.2.1 to cut off the D2-D0 charges at $\beta_i \cdot P < \epsilon P^3$, $|n_i| < \epsilon P^3$, $\epsilon \ll 1$, in order to guarantee existence at least of the first split of the flow tree for any choice of (β_i, n_i) within this cut off domain. This corresponds to the fact that when these charges grow too large, the M2's and anti-M2's associated to the two S^2 poles start interacting so strongly that they can no longer be seen as independent probes and the BPS state can cease to exist altogether. The extreme

[54] This translation was developed in collaboration with Dieter Van den Bleeken [82].

polar state conjecture guarantees we do not need to worry about this for sufficiently polar states (as well as that we do not have to worry about $\mathbb{Z}_r$ quotient geometries).

- Swing states can be seen as BPS states which exist in infinite radius Taub-NUT, but not at some smaller radius — more precisely not in the finite size $AdS_3 \times S^2$ — or vice versa. The example of the swing state of section 6.3.2 can be thought of as a BPS M5 ring which would fit in a Taub-NUT of sufficiently large radius, but not in the finite radius $AdS_3 \times S^2$, as the ring radius becomes too large. At infinite TN radius, this should not be counted as a separate BPS state since all BPS states can be described by light M2 probes which are guaranteed to exist, but at finite radius this is no longer so and BPS states might disappear from the spectrum, hence altering the BPS free energy function which counts BPS states.[55] Somewhat surprisingly perhaps, we found that merely taking the cutoff $\epsilon \ll 1$ is not enough to avoid this phenomenon, but that instead one should take $\epsilon < \delta |P|^{-\xi_{cd}}$ with $\xi_{cd} \geq 1$. If $\xi_{cd} = 1$ (as is the case for our examples and for which we gave some circumstantial evidence at the end of section 6.3.2), the correction to the free energy is of order $\delta\mathcal{F} \sim e^{-g_{\rm top}|P|^2} \sim e^{-(\mathrm{Im}\, t)^2/g}$, indeed suggestive of finite size nonperturbative D4/M5 corrections to the computation of [32, 33]. This can viewed as a further physical indication for $\xi_{cd} = 1$, although we will not try to make this precise here.

- Our restriction to sufficiently strong g arose from the fact that at weak g, the fareytail series ceases to be dominated by the extreme polar terms due to entropic effects. As a result, in this regime, it is no longer justified to disentangle the two sectors, since the main contribution will come from complicated BPS configurations delocalized over the sphere, which do not factorize. The meaning of "strong" and "weak" g depends on the growth of polar indices, to be discussed in the next subsection. This is closely related to the entropy enigma of section 3.5, which in M-theory translates to the entropic dominance of geometries containing two BMPV black holes over the M5 black string.

- The measure factor we find can be traced back to the fact that even in the most dilute gas regime, the M2 and anti-M2 sectors do not fully decouple, since there is a multiplicative contribution to the ground state degeneracies depending on J_R^3 (given by $|\langle\Gamma_1, \Gamma_2\rangle|$ in our setup) which depends on the charges in a non-factorized way. This factor was not taken into account in [17, 18], but was noted in the M-theory context in [47], where it was found necessary for modularity of the M5 elliptic genus and detailed matching with geometrical considerations.

7.4 Range of validity, background dependence and miraculous cancelations

We now turn to the issue of weak versus strong topological string coupling, which as we pointed out depends crucially the growth of polar indices with $|P|$ at fixed η where η

[55]Related to this, as was shown in [98], in certain limits of the IIA Enriques CY moduli space, the BPS free energy is generated by D4-branes indices rather than D2 brane indices, as the former become the light states there.

parametrizes $\hat{q}_0 = (1-\eta)\frac{P^3+c_2 P}{24}$. Note that this is equivalent to the growth of these indices under the scaling $(p^0, p, q, q_0) \to (p^0, \lambda p, \lambda^2 q, \lambda^3 q_0)$ since (in the large λ limit) this leaves η invariant while scaling up $|P| \to \lambda |P|$. Recall furthermore from section 3.3 that this rescaling is a symmetry of arbitrary multicentered BPS configurations (provided one also scales $t_\infty \to \lambda t_\infty$), with corresponding horizon entropy growing as $S \sim \lambda^3$. Thus, this suggests $\log \Omega \sim P^3$ at fixed η, and hence $\kappa = 3$ in (7.13), implying a breakdown of the refined OSV formula (7.1) at weak topological string coupling or equivalently in the limit in which we scale up all charges uniformly.

Now note on the other hand that when the total charge is $(0, P, 0, 0)$, the same non-uniform scaling actually acts uniformly on the *total* charge, so if multicentered configurations exist with nonzero horizon areas and such a total charge, their entropies will scale as λ^3. We saw in section 3.5 that such multicentered solutions do indeed exist, and that moreover a slight extension of the argument just given leads to the conclusion that uniform rescaling of *any* D4-D2-D0 charge (with $P > 0$) by a sufficiently large Λ leads eventually to multicentered configurations with horizon entropy growth $S \sim \Lambda^3$. This suggests $\log \Omega \sim \Lambda^3$ in this regime, in contradiction with the OSV prediction, which scales as Λ^2. This confirms the close relation between the "entropy enigma" of section 3.5 and the breakdown of eq. (7.1), and the fact that the latter is not due to a shortcoming of our derivation itself.

There are two possible loopholes to these conclusions, which could potentially still allow some version of the OSV conjecture to be valid in the large Λ limit. The first one is that we have restricted our attention to a background $t = i\infty$, while perhaps the OSV conjecture should instead be taken to be valid only at some other distinguished point. The second one is that we are considering an index, which gets many contributions with different signs, so there might be miraculous cancelations bringing down the growth of the index compared to the supergravity entropies of individual contributing configurations.

7.4.1 Evaluation point

We do not have much to say about the first possibility. The reason why we considered $t_\infty = i\infty$ only is that our derivation crucially relies on the D4 partition function being a generalized Jacobi-form, and this is only plainly the case at $t = i\infty$. For example, trying to construct some sort of partition function where the indices making up the coefficients are all evaluated at the attractor point of the charge in question would manifestly not give a Jacobi form, since more or less by definition such a "partition function" would not have a polar part (since polar charges do not have attractor points). The necessity to restrict to $t = i\infty$ is not only true for our derivation, but for all derivations based on the fareytail expansion that have appeared [17, 18].

However one could of course contemplate other backgrounds. A natural choice would be to take t_∞ at the attractor point $t_*(p,q)$ of the charge under consideration (assuming this exists), and postulate a formula like

$$\Omega(p,q;t_*(p,q)) \quad \stackrel{?}{\sim} \quad \int d\phi |\mathcal{Z}_{top}(p,\phi)|^2 e^{-2\pi\phi^\Lambda q_\Lambda} \tag{7.19}$$

Unlike the $t = i\infty$ case, there is certainly no evidence against such a claim. For example, the multicentered configurations which lead to the Λ^3 scaling of the entropy do not exist in the attractor background — only configurations encoded by single centered attractor flows, i.e. single centered black holes as well as multicentered "scaling" solutions asymptotically connected to single centered black holes (cf. section 3.8). There are very good physical reasons to believe that, at least to leading order, this version of the OSV conjecture should be correct, since in this background one expects the leading order statistical entropy to be given by the Bekenstein-Hawking-Wald entropy, which coincides with the logarithm of the saddle point value of the right hand side of (7.19).

More generally, one could also consider a fixed finite t_∞ background and take $(p, q) \to \Lambda\,(p, q)$, $\Lambda \to \infty$. From the discussion in section 3.5, one can see that in such a limit, the 2-centered configurations with Λ^3 entropy will again disappear, since the walls of marginal stability move off to infinity when $\Lambda \to \infty$. Note that this raises subtle order of limits issues.

Yet another alternative modifies the OSV conjecture in the form (1.9) by replacing the definition of $\mathcal{Z}_{\rm BH}$ by

$$\check{\mathcal{Z}}_{\rm BH}(p, \phi) := \sum_q \Omega(p, q; t^A = \frac{\Phi^A + iP^A}{\phi^0})\, e^{2\pi\phi^\Lambda q_\Lambda}. \qquad (7.20)$$

This form of the OSV conjecture, regrettably, would seem to be inconsistent with the wall-crossing formulae we have described, so the right-hand side, $|\mathcal{Z}_{\rm top}|^2$ would need to be modified also in some way.

Sadly, a direct microscopic counting at $g_{IIA} = 0$ at finite t_∞ seems out of reach for the charges of interest, because the microscopic description is not sufficiently understood. In the IIA picture, the problem is that α' corrections to the D-term constraints determining the moduli space (in particular determining Π-stability [84, 66]) become manifestly of crucial importance, since they are responsible for the elimination of the "extra" states corresponding to multicentered black holes with Λ^3 entropy growth existing at large radius. These α' corrections are not known systematically. One could try to use mirror symmetry to type IIB, where the relevant D-branes are special Lagrangian 3-cycles and stability becomes just a classical geometrical property, but the problem on this side is (a) the F-terms receive complicated disk instanton corrections and (b) even classically very little is known about special Lagrangian 3-cycles in compact manifolds.

An alternative approach to a derivation, directly at weak $g_{\rm top}$, was suggested in [19], based on AdS-CFT and the computation of the free energy in IIA perturbation theory in a suitable attractor background. There are several points to be clarified in this derivation, and it is not quite a microscopic derivation in the sense of directly counting underlying quantum mechanical degrees of freedom of some brane model. Nevertheless, if correct, the proposal would give a very nice explanation of why $\mathcal{Z}_{\rm top}$ should govern the corrections to the Bekenstein-Hawking entropy.

It would clearly be desirable to know whether the index of D-brane microstates at $t = i\infty$, might still be governed by the OSV formula even in the weak $g_{\rm top}$ regime, so let us next examine the second possibility.

7.4.2 Cancelations between index contributions?

As discussed above, the only way this can happen is if the multicentered black hole entropy for individual configurations corresponding to polar charges grossly overestimates the actual index for a given charge.

Recall that the D4-D2-D0 polar indices are given by the factorization formula (5.4):

$$\Omega(\Gamma, t_\infty) = \sum_{\Gamma \to \Gamma_1 + \Gamma_2} (-1)^{\langle \Gamma_1, \Gamma_2 \rangle - 1} |\langle \Gamma_1, \Gamma_2 \rangle| \; \Omega(\Gamma_1; t_{\rm ms}) \; \Omega(\Gamma_2; t_{\rm ms}) \tag{7.21}$$

where $t_{\rm ms}$ denotes the location of the MS wall for the split $\Gamma \to \Gamma_1 + \Gamma_2$ along the Γ attractor flow starting at $t = i\infty$ (if this exists). As we saw, Γ_1 and Γ_2 need to have nonzero (and of course opposite) D6-charges for the split to exist.

Each of the $\Omega(\Gamma_i; t_{\rm ms})$ could in turn still get contributions from different splittings and possibly also from a single flow (corresponding to a single centered black hole realization of Γ_1), leading to further expressions like

$$\Omega(\Gamma_1; t_{\rm ms}) = \Omega(\Gamma_1; t_*(\Gamma_1)) + \sum_{\Gamma_1 \to \Gamma_1' + \Gamma_2'} (-1)^{\langle \Gamma_1', \Gamma_2' \rangle - 1} |\langle \Gamma_1', \Gamma_2' \rangle| \; \Omega(\Gamma_1'; t_{\rm ms}') \; \Omega(\Gamma_2'; t_{\rm ms}'), \tag{7.22}$$

where $t_*(\Gamma_1)$ is the attractor point of Γ_1, and so on.

Phrased in this framework, the problem we face is that there exist splits $\Gamma \to \Gamma_1 + \Gamma_2$ into two black holes with $S_{BH}(\Gamma_i)$ scaling as $|P|^3$ at fixed η. Identifying $\log \Omega(\Gamma_i, t_*(\Gamma_i)) \approx S_{BH}(\Gamma_i)$, we would thus get a contribution scaling as $e^{c|P|^3}$ to the total index.

Hence we see there are three different ways the total index could still grow more slowly than this:

1. The identification $\log \Omega(\Gamma_i, t_*(\Gamma_i)) \approx S_{BH}(\Gamma_i)$ is wrong, and in fact LHS $\ll$ RHS.
2. There are miraculous cancelations already between the contributions in (7.22).
3. There are miraculous cancelations between the contributions in (7.21).

Possibility (1) is extremely unlikely. In all cases in which one has been able to compute reliably the (proper) index of BPS states of a large black hole at or near its attractor point (e.g. [2, 3, 101]), its logarithm has been found to coincide with the Bekenstein-Hawking-Wald horizon entropy, even beyond leading order. Although physically one expects the horizon entropy to count the true number of BPS states at the attractor point, there is in general no reason to expect this true number to be much larger than the index at finite values of the string coupling, as generically quantum tunneling effects will lift unprotected bose-fermi pairs as soon as the coupling becomes nonzero. (See section 8 for more discussion about this.)

At first sight, a class of five dimensional M2 black holes[56] studied in [3] seems to provide a strong counterexample to this claim: it was found there that the black hole horizon

[56] And therefore, through the 4d-5d correspondence of [70], also a class of four dimensional D6-D2-D0 black holes, for which the D6-brane charge is 1, the D2 charge equals the M2 charge Q, and the D0-charge equals $2J_L^3 = 2m_L$. In other words, precisely the kind of 4d black holes we are interested in here.

entropy matched the total dimension of the cohomology of the corresponding microscopic D-brane moduli space rather than the index, which was taken to be the Euler characteristic, i.e. $N_Q = n_Q^0$ as defined in (6.4) and (6.5). In fact the logarithm of the former was found to grow like $Q^{3/2}$, while the latter grows only like Q, with Q the M2 charge, precisely the kind of miraculous cancelation we are after. However, upon closer inspection, one sees that N_Q is *not* the proper index to compare to. Indeed, a 5d BPS black hole is characterized not only by its M2 charge Q, but also $SU(2)_L$ spin $J_L^3 = m_L$. Therefore, the proper index to compare to is $N_Q^{m_L}$ defined in (6.3) and in fact, we argue in appendix F that for this model, $\log N_Q^{m_L} \sim \sqrt{Q^3 - m_L^2}$, in perfect agreement with the black hole horizon entropy to leading order. The origin of this huge cancelation arising when summing over m_L was explained under (6.5), and can be summarized as $(1-1)^n = 0$. The enhancement of growth going from the $n_Q^r \sim e^Q$ to the $N_Q^{m_L} \sim e^{Q^{3/2}}$ is due to the presence of large binomial coefficients in the relation (6.5) between them, which in turn come from degeneracies due to the Wilson line moduli. In any case, the upshot is that the proper index again agrees with the black hole horizon entropy.

Nevertheless, the cancelation is suggestive. Could it be that in summing over all contributions to our total index, we are effectively summing over m_L (or, in four dimensional language as in footnote 56, over D0-charges), thus producing a near-exact cancelation? This brings us to possibilities (2) and (3) in the list above. In particular, as we will see below, possibility (2) might be related to this.

Before we go on, it is worth emphasizing that in general cancelations changing the exponential growth behavior have to be pretty miraculous indeed. Consider for simplicity two contributions of nearly the same size but with opposite signs, say

$$\zeta := e^{c|P|^3} - e^{(c+\epsilon)|P|^3} \approx \epsilon |P|^3 e^{c|P|^3}. \tag{7.23}$$

Then to get $\zeta \sim e^{c'|P|^2}$, we need ϵ to be of order $e^{-c|P|^3}$! So it is hard to imagine a significant cancelation in our index unless all leading order contributions cancel exactly, and only exponentially subleading contributions remain.

In view of this, it seems highly unlikely that such cancelations could occur as described in possibility (3). Moreover, even if there were such a cancelation for contributions at $t = i\infty$, there would not be such a cancelation at other values of t, since different splits $\Gamma \to \Gamma_1 + \Gamma_2$ have different walls of marginal stability, so if there were cancelation at one point in moduli space, this would almost certainly not hold at some other point, since the set of contributing splits will be different at these two points. Thus, if cancelation is to happen, one would expect it to take place already at the level of the contributions to $\Omega(\Gamma_1)$ and $\Omega(\Gamma_2)$, i.e. possibility (2).

Unfortunately, possibility (2) also appears highly unlikely. In this case, we can relate the problem, to some extent, to a precise mathematical question about the asymptotic growth of DT invariants. Recall that according to (6.78) and (6.21)-(6.24), we have, at least for $\eta < \eta_* \ll 1$,

$$\Omega(\Gamma_i, t_{\rm ms}) = N_{DT}(\beta_i, n_i) + \Delta\Omega(\Gamma_i, t_{\rm ms}) \tag{7.24}$$

with (β_i, n_i) defined as in (4.17)-(4.19). As we showed in section 6.3.2, $\Delta\Omega$ can be nonzero due to swing states. By definition, for $\eta < \delta |P|^{-\xi_{cd}}$ (with $1 \leq \xi_{cd} \leq 3$), we have $\Delta\Omega = 0$, but in the case at hand we cannot use this since we want to keep η fixed while scaling up P, so we always exit this regime. Thus we expect $\Delta\Omega \neq 0$.

As we will see below, the generating function for DT invariants has some special structure which allows one to make some (very) heuristic arguments in favor of miraculous cancelation of different contributions to the DT invariants. If such cancelations indeed occur, then it would become perhaps less implausible that something like it might happen for $\Omega(\Gamma_i, t_{\rm ms})$ as well. However, note that this is far from obvious. First, for non-extreme polar states, i.e. values of η closer to 1, there will also be contributions from rank $r > 1$ D6-anti-D6 splits, which are not directly related to the above DT invariants. Second, the individual flow tree contributions to *each* of the two terms on the right hand side of (7.24) will give contributions to the index scaling as e^{λ^3} under the large λ scaling $(p^0, p, q, q_0) \to (p^0, \lambda p, \lambda^2 q, \lambda^3 q_0)$ we are considering (this acts on the (β_i, n_i) parameters as $(\beta_i, n_i) \to (\lambda^2 \beta_i, \lambda^3 n_i)$), as follows from the general arguments of section 3.3. Hence we would need miraculous cancelations between the contributions to $\Delta\Omega$ separately as well. Moreover, by varying (β_2, n_2) while keeping (β_1, n_1) fixed, $t_{\rm ms}$ will vary, and so $\Delta\Omega$ can change if this variation takes $t_{\rm ms}$ over some marginal stability wall. Again individual contributions to this variation of $\Delta\Omega$ scale as e^{λ^3}, so cancelation should occur already within this very reduced ensemble. It seems hard to imagine how something like this could happen unless there is extended supersymmetry killing off individual contributions in bose-fermi pairs.

Nevertheless, since the problem for DT invariants can be formulated in a mathematically precise way, and the question is of some interest on its own, let us proceed to investigate the scaling

$$\log N_{DT}(\lambda^2 \beta, \lambda^3 n) \sim \lambda^k, \tag{7.25}$$

and ask whether $k = 3$, as suggested by the scaling argument, or whether cancelations occur making $k \leq 2$. (Note that the asymptotic growth $N_{DT}(0, n) \sim e^{n^{2/3}}$, given in (6.2), suggests that we always have at least $k \geq 2$.[57])

To be more precise, consider the limit-supremum:

$$k = \overline{\lim}_{\lambda \to +\infty} \frac{\log \log |N_{DT}(\lambda^2 \beta, \lambda^3 n)|}{\log \lambda} \tag{7.26}$$

The first question is whether this is independent of (β, n), and hence equal to a constant k. We expect this to be the case. If this is indeed so, then the next and crucial question is the value of k. Unfortunately, the answer to this question seems to be unknown. Even for particular cases, we have been unable to find k. The reason is that to compute the DT invariants to sufficiently high order, one needs BPS invariants n^r_Q to sufficiently high

[57] Except when $\chi(X) = 0$. Indeed, for T^6 and $T^2 \times K3$, this entire discussion is superfluous: for T^6, $\mathcal{Z}_{\rm DT} = 1$, and for $T^2 \times K3$, $\mathcal{Z}_{\rm DT} = \eta(t)^{-24}$ where t is the T^2 Kähler modulus. In other words, in these cases we *do* have "miraculous" cancelations, although in this case the miracle is simply extended supersymmetry.

order Q, and despite tremendous recent progress [102], the available data so far are not yet sufficient to get even a numerical hint of what the correct answer could be. [58]

Let us therefore turn to a some heuristic arguments - two pro and two contra - for the cancellation hypothesis.

1. The first heuristic argument suggesting $k = 2$ goes as follows. What we want to compute is
$$N_{DT}(\lambda^2\beta, \lambda^3 n) \sim \oint dt\, dg\, e^{gn\lambda^3 - 2\pi i t\cdot\beta\lambda^2}\, \mathcal{Z}_{DT}(-e^{-g}, e^{2\pi i t}). \tag{7.27}$$
Now, because of the DT-GW correspondence as reviewed in section 1.3, we can formally write, at small g,
$$Z_{DT}(-e^{-g}, e^{2\pi i t}) \sim e^{\frac{1}{g^2} f(t)} \tag{7.28}$$
where $f(t)$ is the generating function for genus zero Gromov-Witten invariants. Plugging this in (7.27) and doing a naive saddle point evaluation gives saddle point equations of the form
$$n \sim \frac{f(t)}{g_0^3}, \qquad \beta \sim \frac{f'(t)}{g_0^2}, \qquad g =: g_0/\lambda, \tag{7.29}$$
and saddle point value
$$N_{DT}(\lambda^2\beta, \lambda^3 n) \sim e^{c\lambda^2}, \tag{7.30}$$
where c is independent of λ. In other words, this indeed suggests $k = 2$, hence miraculous cancelations! On the other hand, it is not clearly valid to use (7.28) in such a saddle point analysis, so the argument is only heuristic.

2. To be conclusive, one would like to see the cancelations happening directly. To see where these might come from and how they might be related after all to the cancelations we mentioned earlier in the context of 5d M2 black holes, recall the expression (6.7)-(6.11) for $\mathcal{Z}_{DT}$. Note that when $g \to 0$, $u \equiv -e^{-g} \to -1$ and $\mathcal{Z}_{DT}^{\prime, r>0} \to \prod_q (1 - v^q)^{-n_q^1}$. This collapse is due to the same kind of binomial coefficient cancelations we saw in the 5d black hole context and as explained under (6.5), killing all $r > 1$ contributions. Of course this is as expected from the DT-GW correspondence: only the genus zero and one contributions to $F_{\rm top}$ survive when $g \to 0$. But it is noteworthy that these "miraculous cancelations" are intimately tied together with this correspondence, indicating that the heuristic result we found does have something to do with the existence of cancelations.

   Furthermore, on general grounds, and as suggested by the heuristic argument, it is conceivable that the large λ scaling of (7.25) is governed by the $g \to 0$ behavior of

[58] There are some examples where one could extract k on compact Calabi-Yau manifolds. These use results on F_g to all orders derived from heterotic/typeIIA duality, [97, 34, 98]. However, in precisely these cases the relevant black holes have $P^3 = 0$. Thus, one should suspect that the case where β represents a holomorphic curve in the K3 fiber of a K3-fibered Calabi-Yau is not representative. That is, there is in fact some dependence of (7.26) on β and really we should be asking about *generic* β.

$\mathcal{Z}_{DT}$, i.e. the $r = 0$ factor and to a lesser extent the $r = 1$ factor. Now if we drop the $r > 1$ factors, we have effectively dropped our reasons to expect $k = 3$ in (7.25), since as we noted in our discussion of 5d black holes above, the enhancement of growth going from the $n^r_Q \sim e^Q$ to the $N^{m_L}_Q \sim e^{Q^{3/2}}$ is due to the presence of large binomial coefficients in the relation (6.5) between them, but if we drop all $r > 1$ contributions, these are no longer present.

Although this fortifies the case for cancelations, it is still not conclusive. We did some numerical experiments with very simple toy models for which naively one could make the same reasoning as above, but which nevertheless did not lead to any cancelations. Unfortunately, as we noted before, not enough hard data about the BPS invariants n^r_Q is available at this time to check these arguments by direct computation in an actual (compact) model.

3. An argument against the cancelation hypothesis follows if we assume the OSV conjecture holds for $p^0 = 1$, for some value of the B-field at $J = \infty$. Then $N_{DT}(\lambda^2\beta, \lambda^3 n)$ should be given by the OSV formula, which for $\lambda \to \infty$ predicts a growth $N_{DT} \sim \exp[\lambda^3\sqrt{\beta^3 - n^2}]$, leading to $k = 3$. Indeed, one concrete conclusion from these considerations is that the OSV conjecture for $p^0 = 1$, $t_\infty \to i\infty$ and the weak coupling OSV conjecture for $p^0 = 0$, $t_\infty \to i\infty$ cannot both be true.

4. We conclude by giving a second heuristic argument indicating there are no significant cancelations, so that the entropy enigma is also an index enigma, i.e. that indeed $\log\Omega(\Lambda\Gamma; t = i\infty)$ does grow as Λ^3 in the large Λ limit, with Γ some D4-D2-D0 charge.[59] For simplicity we take $\Gamma = (0, P, 0, 0)$. A naive model for this is a D4-brane wrapped on P with $N = \chi(P)/24 \approx P^3/24$ pointlike $\overline{D0}$-branes bound to it. Ignoring divisor moduli, flux degrees of freedom and so on, the index of this system is simply the orbifold Euler characteristic of the N-fold symmetric product of P. This is given by the coefficient d_N of q^N in $\prod_n (1 - q^n)^{-\chi(P)}$. The $N \to \infty$ asymptotics are given by the Cardy formula $\log d_N \sim 4\pi\sqrt{(N - \frac{\chi}{24})\frac{\chi}{24}}$, and this equals the single centered entropy for this charge. However, the value $N = \chi/24$ of interest to us lies outside the regime of validity of this formula — in fact plugging this in the formula gives zero. Numerically on the other hand, we find $\log d_{N=\chi/24} \approx 0.17649134 * \chi \sim P^3$. (Exponential growth of this coefficient can also be proved analytically.) Note that this is the same growth as suggested by the two-centered black hole estimate without cancelations! But again this argument is too heuristic to be taken seriously; in particular, although this model for the D4-D0 system is fine in the limit $N \to \infty$, it is not clearly applicable to the case $N \sim P^3$, since there is now no justification for ignoring the divisor moduli and fluxes.

In conclusion, although we cannot completely exclude a miracle, it seems very unlikely to us that a sufficient amount of cancelation could occur to bring down the polar index growth at fixed η from $\sim e^{P^3}$ to $\sim e^{P^2}$, and we therefore believe $\kappa = 3$ in (7.13), hence

[59] We thank D. Gaiotto, A. Strominger and X. Yin for a related suggestion leading to this argument.

a breakdown of OSV at weak $g_{\rm top}$ (and $t_\infty = i\infty$). It would be interesting to settle this question definitively.

7.5 Dumping the dangerous swing states

We now briefly return to the second main unresolved issue, namely the value of the core dump exponent ξ_{cd} introduced in section 6.3.2.

At the end of section 6.3.2 we offered some circumstantial evidence suggesting that perhaps $\xi_{cd} = 1$, and all is well. In addition to this, we can offer the following physical evidence that suggests that $\xi_{cd} = 1$. As noted in section 7.3, in this case, the D4-D2-D0 partition function differs from $|\mathcal{Z}_{\rm top}|^2$ by terms whose order indicates that they involve Schwinger pair production of wrapped D4-brane states, giving finite size corrections to the exactly factorized expression. Analogous states have been seen to play a role in topological string amplitudes before [98]. Since there are no other obvious physical effects which would be larger, one might hope that $\xi_{cd} = 1$.

Clearly it is of great interest to investigate these phenomena further to see whether or not the core-dump exponent ξ_{cd} is larger than one. It should be relatively straightforward to come up with further systematic analytical and numerical evidence, similar to the evidence we accumulated in favor of the extreme polar state conjecture, but we leave this for future work.

One of the reasons that the OSV conjecture is interesting is that it suggests a way to give a nonperturbative definition to the topological string. In view of this it is intriguing that the corrections we find are indeed suggestive of nonperturbative corrections. Therefore it might be useful to ask how to compute the contribution of these nonperturbative effects to F-terms in effective supergravity.

8. Summary of open problems and potential future directions

In this section we collect and summarize the many issues and open problems which arose in our derivation of the OSV conjecture. We also suggest some potentially interesting future directions for research.

First, as already discussed in section 1.2, our "proof" of the OSV conjecture is really more of an outline for a proof. The following important issues need to be settled before the argument truly constitutes a proof, even in the strong coupling regime:

- Some basic issues in the theory of split attractor flows and multicentered black hole solutions remain to be clarified. While physically very well motivated, and supported by numerous examples, the split attractor flow conjecture of section 3.2.2 remains to be proven mathematically. Moreover, as we discussed, the Hilbert space of BPS states $\mathcal{H}(\Gamma; t)$ is — roughly speaking — "graded" by the split attractor flows associated to $(\Gamma; t)$, but we noted some subtleties, and hence the precise rule remains to be elucidated. Among other things one should understand better the possible quantum mixing between states associated to different attractor flow trees.

- It would be desirable to have a more systematic derivation of the D4 partition function we used in section 2 starting from a path integral. Ideally, this should clarify the relation of $d(F, N)$ to general DT invariants.

- The extreme polar state conjecture remains to be proved.

- We have shown that certain swing (core) states could potentially invalidate the OSV conjecture. This led to the definition of the core dump exponent in section 6.3.2. As discussed there, it remains to show that $\xi_{cd} \leq 2$. If it turns out that $\xi_{cd} > 2$ then the OSV conjecture is very unlikely to be true, even at strong $g_{\rm top}$. We outlined some indications that $\xi_{cd} = 1$, but further work is needed to test this hypothesis.

- The equality of DT and Gromov-Witten partition functions for compact CY remains to be proved.

Certainly, at the "physical level of rigour" the extreme polar state conjecture and the claim that $\xi_{cd} \leq 2$ are the main gaps in our argument. We are rather confident that all the above issues – be they ever so challenging – can be satisfactorily settled, with the possible exception of $\xi_{cd} \leq 2$. Granting these points, there are a number of ways in which the refined OSV formula could be extended further.

- First there is the question of the extension from strong to weak topological string coupling. As we have demonstrated, this is loosely related to a well-posed question regarding asymptotics of DT invariants, namely, the evaluation of

$$k = \overline{\lim}_{\lambda \to \infty} \frac{\log \log |N_{DT}(\lambda^2 \beta, \lambda^3 n)|}{\log \lambda}. \tag{8.1}$$

  However, while an anomalous value $k \leq 2$ instead of the expected $k = 3$ would certainly be suggestive, this would not immediately imply a similar growth of the relevant indices $\Omega(\Gamma_i, t_{\rm ms})$. Regarding the latter, we see little hope of a cancellation and expect that the OSV conjecture fails at weak $g_{\rm top}$. It would however be very interesting to verify this more directly.

- As we saw, the "core states" account for the genus $r > 0$ component of the Gopakumar-Vafa product form of the topological string partition function. While some core states are black holes, there are also more complicated core states. It would be very interesting to find some way to organize and classify these core states. It would seem that this is essential to evaluating ξ_{cd}.

- Can our methods be extended to values of P which are not in the Kähler cone but nevertheless support BPS states? (Consider, for example, a curve of resolved ADE singularities.)

- The original paper [10] claimed a version of the conjecture for *all* magnetic charges, including $p^0 \neq 0$. The wall-crossing formulae we have discussed would seem to pose a serious obstacle for such a version of the conjecture, at least if it is based on the degeneracies at $\operatorname{Im} t = \infty$. Is there nevertheless a version for $p^0 \neq 0$?

Let us now turn to various questions and potential physical applications which our paper raises:

- Our work sheds some light on the old confusion of the relevance of absolute BPS degeneracies versus indices. On physical grounds, one expects the total BPS Bekenstein-Hawking-Wald entropy and refinements thereof to correspond to the absolute number of BPS states. Naively one might therefore think one should compare the total dimension of the cohomology of the relevant D-brane moduli spaces rather than the euler characteristic, which is the index. However, one should keep in mind that the quantum mechanics of D-brane moduli spaces is only a low energy approximation to the true physical situation. The effective quantum mechanics ignores some of the degrees of freedom on the D-brane. In particular, one should take the string coupling constant to be zero. As soon as it is nonzero, instanton effects come into play. For example there might be tunneling between different classical supersymmetric ground states, i.e. between different components of the D-brane moduli spaces. In particular we expect D2-instantons tunneling between different flux sectors with the same total charge, producing effects of order $\sim e^{-\sqrt{J^3}/g_{\rm IIA}} \sim e^{-1/g^{4d}_{\rm IIA}}$.[60] These are external to the moduli space quantum mechanics, and will generically lift bose-fermi pairs of supersymmetric ground states of the latter. Based on genericity, one could therefore reason that in fact, all nonprotected states can be expected to be lifted, bringing the total degeneracy down to the value of the index.

  There is quite a bit of evidence in favor of this idea. First, the detailed agreement we find in this paper (and the agreement found in related work) is with the index, not with the total cohomology of moduli space. Indeed, the BPS invariants n^r_Q determining $\mathcal{Z}_{\rm DT}$ are all indices, not total betti numbers. The simplest example of this is the power of the MacMahon factor, which is the euler characteristic $\chi(X)$ of the Calabi-Yau, not its total cohomology. It should furthermore be noted that these index invariants appear already in the leading order supergravity entropy formula, even neglecting R^2 and higher order curvature corrections — for example the contribution proportional to $\zeta(3)\,\chi(X)$ in the IIA supergravity entropy formula is obtained from the MacMahon factor counting indices as in (1.16). This indicates that subtleties involving higher curvature corrections (and the fact that we are comparing to the BHW entropy taking into account only F-term R^2 corrections) are largely irrelevant for this discussion.

  In fact it has been known for quite some time [8] that D4-D0 black holes for $T^2 \times K3$ and T^6 compactifications have, already in the large D0-charge limit, BHW entropies which do not match the total degeneracy at $g_{\rm IIA} = 0$, while they agree with the index (this discrepancy did not arise for $SU(3)$ holonomy Calabi-Yau compactifications because the D4-D0 brane moduli spaces were approximated as N-fold symmetric products of a very ample divisor, which in the $SU(3)$ holonomy case does not have

[60] Here $g^{4d}_{\rm IIA}$ is the effective four dimensional string coupling, which sits in a hypermultiplet and is therefore tunable without affecting any BPS indices.

any odd cohomology, making the index equal to the total degeneracy). The class of 5d black holes studied in [3] seemed to go the other way, but as we show in appendix F, also in this case there is again exact agreement with the index, provided the proper index is used.

In conclusion, we see considerable evidence that the true number of BPS states at finite coupling is in fact given by the microscopic index.[61] If the total number of BPS states at zero coupling is higher than the index, we expect a number of states with energies no more than $\Delta E \sim e^{-1/g_{IIA}^{4d}}$ above the BPS bound.

For an in depth recent discussion of related issues in the case of extremal nonsupersymmetric black holes see [103].

On the supergravity side, we have a parallel picture. Since there are in general many alternating sign contributions to the BPS index, possibly coming from many different multicentered configurations, there is again ample room for tunneling effects to lift BPS states obtained in the supergravity moduli space approximation. For example in our D6-anti-D6 description of polar states, an anti-D2 particle in a halo around the D6 could annihilate with a D2 particle in a halo around the anti-D6. Following the heuristic dictionary of section 4.3, such a process should be the supergravity analog of a D2-instanton tunneling between different flux sectors.

Thus, on both sides, in suitable coupling regimes, this suggests a picture of having essentially $\log |\Omega(\Gamma; t_\infty)|$ exact BPS ground states of charge Γ in a background t_∞, as well as a certain number of slightly non-BPS states at exponentially small energies above the BPS bound. Now, this number might actually be quite huge: indeed if there is a cancelation $e^{\Lambda^3} \to e^{\Lambda^2}$ in the index of the kind discussed at length in section 7.4.2, and if indeed all canceling contributions in the index get lifted by tunneling effects, then the number of near-BPS states with exponentially small energy gaps would in fact be of order e^{Λ^3}, dwarfing the number of exact BPS states! Even if there is no such cancelation, one would still expect a comparable amount of bosonic and fermionic states, and therefore $\sim e^{\Lambda^3}$ non-BPS states with exponentially small energy gaps.

It would be interesting to see to what extent these speculations are correct, and if so, if perhaps there might be implications for models of dynamical supersymmetry breaking.

- It would be helpful to elucidate the connection between our approach and the approach based on the MSW conformal field theory. The MSW CFT is the effective conformal field theory with $(0,4)$ supersymmetry describing a string obtained from

[61] We should mention though that in [11] (section 4) the BHW entropy of a class of small black holes with untwisted sector charges in the FHSV model was found to agree with the absolute cohomology to leading order, but not with the index (while in the twisted sector the index did agree). It is conceivable however that in this case there exists again a more refined index which would also be in agreement in the untwisted case. Alternatively, certain Kähler classes were set to zero for these small black holes, and perhaps the background is too singular.

wrapping an M5 brane on a holomorphic surface such as P [2, 105]. The precise formulation of this CFT remains incomplete. Assuming the formulation can be completed, it would be interesting to clarify the relation between our D4 partition function and the elliptic genus of the MSW CFT.

- In equation (5.49) we found a remarkably simple formula for the entropy associated with a 3-node quiver with a loop. The challenge remains to find a conceptual derivation of this formula. Will it extend to other quivers with loops? Conceivably, there might be interesting applications to the AdS/CFT correspondence.

- Our discussion of multicentered black holes and their moduli spaces should have implications for the "Mathur program," which aims to account for microstate entropy from quantizing supergravity solution moduli spaces [115]. In particular in [79] it was proposed that quantizing moduli spaces of horizon-free four dimensional multicentered configurations with given total charge might account for the corresponding black hole entropy. The results we found for the three node quiver in sections 5.2.3 and 5.3 seem to create some tension with this proposal. While in the regime in which there are no scaling solutions to the integrability constraints, the microscopic ($g_{\rm IIA} \to 0$) quiver degeneracies are correctly reproduced from quantizing the associated three centered system, it seems rather unlikely that this remains the case in the regime in which there are scaling solutions, given the exponential growth (5.49) of the microscopic degeneracies in this regime and the relative simplicity of the corresponding three particle quantum mechanics. Although we can't exclude that the set of *all* multicentered configurations with the same total charge might still add up to the same degeneracy as the set of all microscopic realizations with the same total charge, in our opinion, our result is instead rather suggestive of the appearance of new degrees of freedom and BPS configurations in the scaling regime, qualitatively different from the multicentered configurations considered so far. (In [79] are related suggestion of the appearance of nonabelian degrees of freedom was made, based on $SU(N)$ degrees of freedom associated with nodes of the quiver. The striking thing about the present example is that the nodes are associated with rank one gauge groups.)

- In this setting, an open problem remains. In the split attractor flow conjecture the scaling solutions are in the same component of moduli space as a simple single-centered flow for the total charge, and hence are represented by the same flow. These scaling solutions cannot be forced to decay by changing the background moduli. We do not know however if all multicentered scaling solutions to the integrability conditions (3.21) with the total charge of a black hole in fact correspond to actual BPS solutions of supergravity, and if not, whether one can significantly simplify the rather cumbersome criterion for existence: $\mathcal{D}(H(\vec{x})) > 0$ for all $\vec{x} \in \mathbb{R}^3$.

- The appearance of the measure factor (6.115) in the integral form of the OSV formula (6.113) is striking and very reminiscent of Kähler quantization. In this interpretation the black hole degeneracies are certain kinds of Wigner functions for a distinguished quantum state provided by the topological string. Of course, this observation has been made before [113, 114], but there have been some difficulties making this proposal precise. (For example, the topological string partition function is not a normalizable wavefunction. In fact it is not even a function, since it has zero radius of convergence.) We hope our precise version can help clarify this conjecture.

- Our results should have several model building applications, since, at least classically, the conditions to have supersymmetric brane configurations are independent of whether the branes are space-filling or not. One issue that generally has been ignored in phenomenological D-brane model building is stability. For example D7 branes wrapping divisors in IIB orientifold models might well decay when the volume of the divisor gets too "small," just like the corresponding D4-branes in type IIA. But as we saw, small is not all that small actually if the divisor has a large Euler characteristic. Moreover, flux compactifications with all moduli stabilized typically do not have parametrically large divisor volumes. Thus, stability becomes quite relevant, and the practical tools we developed here might be useful to get a handle on this.

Finally, we think there are mathematical implications and applications which could be of some interest.

- If indeed the degeneracies $d(F, N)$ are DT invariants then our main claim is that these can be arranged in an interesting modular generating function, and our factorization formulae imply highly nontrivial polynomial relations between the DT invariants.

- Do the physical results here, especially the picture of section 4.3, shed light on the geometry of the Noether-Lefshetz locus?

- In the discussion of the fareytail expansion a certain interesting class of polynomials h_γ arose. What are these polynomials? Do they have a physical interpretation? What is the analog when $|w_H|$ is half-integral?

- In equation (5.6) we derived a wall crossing formula. However, it is incomplete since we should like to account for all nonprimitive splits $\Gamma \to N_1\Gamma_1 + N_2\Gamma_2$ across a wall of marginal stability. Generalizing to the case where N_1, N_2 are both greater than one appears to be an interesting and challenging problem.

- It would be interesting to clarify the relation of our framework to the study of Π-stable objects in the derived category of coherent sheaves on X. It has not escaped our attention that D. Joyce has also arrived at wall-crossing formulae reminiscent of ours from a very different point of view (see [68] and references therein). It is clearly of value to examine the relation between these formulae.

- There is an old idea that there should be an interesting algebraic structure associated with BPS states of D-branes on Calabi-Yau manifolds [117, 88, 118]. Indeed, infinite products such as (6.8)-(6.11) are reminiscent of denominator products associated to generalized Kac-Moody algebras. The algebra should be graded by $K(X)$, and should depend on the moduli t_∞. Along lines of marginal stability where $\Gamma \to \Gamma_1 + \Gamma_2$ we should associate algebra products $\mathcal{H}(\Gamma_1; t) \otimes \mathcal{H}(\Gamma_2; t) \to \mathcal{H}(\Gamma; t)$. These algebras should generalize the geometric construction of highest weight modules of affine Lie algebras due to Nakajima. Perhaps some of the examples and techniques of this paper could be usefully applied to realizing this dream.

- As we have mentioned, in the dual M-theory viewpoint the BPS indices are computed using the elliptic genus of the MSW CFT [17, 47, 18]. It is interesting to ask how the wall-crossing formula should arise in this context. Studies of the $(0,4)$ elliptic genus thus far have been limited to the region $t_\infty = i\infty$, but presumably there is an extension of the $(0,4)$ CFT and its elliptic genus to finite values of t_∞. In this context many interesting new issues will arise. First of all, the $(0,4)$ CFT is rather subtle due to the discriminant locus in the moduli space of the surface on which the M5 is wrapped, and hence the very definition of the Dirac-Ramond operator will be subtle. Next, one may guess that as one continues the background moduli through certain walls the Dirac-Ramond operator fails even to be formally Fredholm and its character-valued index can change. It would be very interesting to recover the wall-crossing formulae from this point of view.

- In section 6.1.2 we showed that there is a difference between D6D2D0 degeneracies and DT invariants. Only in a special limit of the B-field do they correspond. This raises the question of what the general relation is, and whether other limits of the B-field could be taken. (That in turn reduces to detailed applications of wall-crossing formulae which we have not tried to sort out.)

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A. Definitions and conventions

In the IIA picture, we write charges $\Gamma \in H^{\text{even}}(X)$ in components as

$$\Gamma = \Gamma^0 + \Gamma^A D_A + \Gamma_A \widetilde{D}^A + \Gamma_0\, \omega \tag{A.1}$$

where $\{D_A\}$ is a basis of $H^2(X,\mathbb{Z})$, $\{\widetilde{D}^A\}$ is a dual basis, and ω is the unit volume element on X, dual to 1. The index A runs from 1 to $h^{1,1}(X) := h$. When Γ is identified with (p,q), we have $p^\Lambda = \Gamma^\Lambda$, $q_\Lambda = \Gamma_\Lambda$. The index Λ runs from 0 to h. Quantities with an A index tend to get capitalized. Note that $\Phi^A = \phi^A$, $P^A = p^A$ etc.

The Dirac-Schwinger-Zwanziger symplectic intersection product is defined as

$$\langle \Gamma, \Delta \rangle = -\Gamma^0 \Delta_0 + \Gamma^A \Delta_A - \Gamma_A \Delta^A + \Gamma_0 \Delta^0 = \int_X \Gamma \wedge \Delta^*, \tag{A.2}$$

where Δ^* is obtained from Δ by inverting the sign of the 2- and 6-form components. When Γ_1 and Γ_2 are represented as sheaves V_1 and V_2, then their charges are given by

$$\Gamma_i = \mathrm{ch}(V_i)\,\sqrt{\widehat{A}(X)} = \mathrm{ch}(V_i)\,(1 + \frac{c_2(X)}{24}) \tag{A.3}$$

and, by the Grothendieck-Riemann-Roch theorem,

$$\sum_k (-1)^k \,\dim \mathrm{Ext}^k(V_1, V_2) = \langle \Gamma_1, \Gamma_2 \rangle = \int_X \mathrm{ch}(V_1)\,\mathrm{ch}(V_2^*)\,\widehat{A}(X). \tag{A.4}$$

The normalized period vector is defined as

$$\Omega_{\mathrm{nrm}}(t,\bar{t}) := e^{\frac{1}{2}K(t,\bar{t})}\,\Omega_{\mathrm{hol}}(t) \;\in H^{2*}(X,\mathbb{C}), \qquad K := -\ln i \langle \Omega_{\mathrm{hol}}, \overline{\Omega_{\mathrm{hol}}} \rangle, \tag{A.5}$$

which depends on the complexified Kähler moduli fields $t^A = B^A + iJ^A$ and in the large radius approximation is given by

$$\Omega_{\mathrm{hol}} = -e^{B+iJ}, \qquad \Omega_{\mathrm{nrm}} = \frac{\Omega_{\mathrm{hol}}}{\sqrt{4J^3/3}}. \tag{A.6}$$

From this, one defines the holomorphic central charge

$$Z_{\mathrm{hol}}(\Gamma, t) := \langle \Gamma, \Omega_{\mathrm{hol}}(t) \rangle, \tag{A.7}$$

which in the large radius approximation becomes

$$\begin{aligned} Z_{\mathrm{hol}}(p,q;t) &= -\int_X (p^0 + P + Q + q_0\,\omega) \wedge e^{-t} && \text{(A.8)} \\ &= p^0 \frac{t^3}{6} - P \cdot \frac{t^2}{2} + Q \cdot t - q_0 && \text{(A.9)} \\ &= \frac{1}{6} p^0 D_{ABC} t^A t^B t^C - \frac{1}{2} p^A D_{ABC} t^B t^C + q_A t^A - q_0 && \text{(A.10)} \end{aligned}$$

where $D_{ABC} := D_A \cdot D_B \cdot D_C := \int_X D_A \wedge D_B \wedge D_C$. The normalized central charge is obtained from this as $Z_{\mathrm{nrm}} = e^{\frac{1}{2}K} Z_{\mathrm{hol}}$. We usually drop the subscript distinguishing holomorphic and normalized central charges when no confusion can arise. Sometimes we abbreviate $Z(\Gamma;t)$ to $Z(\Gamma)$.

B. Some algebraic geometry

We collect here some mathematical facts used in the text. Many of these are nicely explained and were skillfully applied in the present context in [2]. Many of the mathematical facts can be found explained in detail in [104], chapter 1.

Let X be a projective variety. A divisor class is determined by $P \in H^2(X, \mathbb{Z})$. If P is of type $(1,1)$ it is the first Chern class of a holomorphic line bundle $\mathcal{L}_P$, and effective divisors in the divisor class are vanishing loci of sections of $\mathcal{L}_P$. The moduli space of these divisors is a projective space $\mathcal{M}_P = \mathbb{P}H^0(X, \mathcal{L}_P)$, called a *complete linear system*, also denoted $|P|$. The generic divisor in $\mathcal{M}_P$ is a smooth hypersurface in X, but there is a discriminant locus $\mathcal{D}$ in $\mathcal{M}_P$ of singular divisors.

For example, if X is the quintic $\sum X_i^5 = 0$ in $\mathbb{P}^4$ then $P = nH$ where $n > 0$ is integral and H is the Kähler class of $\mathbb{P}^4$ and the linear system $\mathcal{M}_P$ consists of the set of divisors defined by the vanishing of a degree n polynomial on $\mathbb{P}^4$ intersected with $\sum X_i^5 = 0$. The discriminant locus $\mathcal{D}$ is already quite complicated for $n = 1$. In this case the divisors are defined by $\sum \alpha_i X_i = 0$ with $[\alpha_1 : \cdots : \alpha_5] \in \mathbb{P}^4$ and the discriminant locus is defined by $\alpha_i = b_i^4$ where $[b_1 : \cdots : b_5] \in X$.

The dimension of the moduli space $\dim \mathcal{M}_P$ can, under some circumstances be obtained by combining the index theorem with vanishing theorems. The index theorem says

$$\sum_i (-1)^i h^i = \int_X e^P \, \mathrm{Td}(T^{1,0}X) \tag{B.1}$$

where $h^i = \dim H^i(X, \mathcal{L}_P)$. Now, P is *very ample* iff the sections $s : X \to \mathbb{P}H^0(X, \mathcal{L}_P)$ define an embedding ([104], p. 192). On the other hand, P is said to be *ample* if some positive multiple of it is very ample. A criterion for being ample is that P is positive as a (1,1)-form, i.e. $P_{i\bar{j}} > 0$, which in turn is true iff P lies within the Kähler cone, i.e. $\beta \cdot P > 0$, $D \cdot P^2 > 0$, $P^3 > 0$ for all effective curves β and divisors D. In this case $h^i(\mathcal{L}_P) = 0$ for $i > 0$ ([104] p.154), and $\mathcal{M}_P$ is just a projective space of complex dimension $h^0 - 1$ which can be read off from B.1.

Specializing to a Calabi-Yau 3-fold X we have $\mathrm{Td}(T^{1,0}X) = 1 + c_2(X)/12$ and hence

$$\dim \mathcal{M}_P = \frac{1}{6}P^3 + \frac{1}{12}Pc_2(X) - 1 \tag{B.2}$$

and hence $\chi(\mathcal{M}_P) = \frac{1}{6}P^3 + \frac{1}{12}Pc_2(X)$.

As explained in detail in [2], using the Hirzebruch signature theorem and $\chi(\Sigma) = \int_\Sigma c_2(T\Sigma)$ together with the adjunction formula we get

$$\chi(\Sigma) = P^3 + Pc_2(X) \tag{B.3}$$

$$\sigma(\Sigma) = -\frac{1}{3}P^3 - \frac{2}{3}Pc_2(X). \tag{B.4}$$

We often denote $\chi(\Sigma)$ by $\chi(P)$. Now eqs. (B.3),(B.4) in turn imply

$$b_2^+(\Sigma) = \frac{1}{3}P^3 + \frac{1}{6}Pc_2(X) + b_1(\Sigma) - 1 \tag{B.5}$$

$$b_2^-(\Sigma) = \frac{2}{3}P^3 + \frac{5}{6}Pc_2(X) + b_1(\Sigma) - 1. \tag{B.6}$$

Note that for "large" P (such as we consider in this paper) the topology of Σ is quite complicated. For example, on the quintic, if $P = nH$ we have $\chi(\mathcal{M}_P) = \frac{5n^3+25n}{6}$ (an integer) and $\chi(\Sigma) = 5n^3 + 50n$.

In the text we use the Lefschetz Hyperplane theorem ([104], p. 156) which guarantees for very ample Σ that the pullback $H^q(X, \mathbb{Q}) \to H^q(\Sigma, \mathbb{Q})$ is an isomorphism for $q \leq \dim X - 2$ and is injective for $q = \dim X - 1$. It follows that, if X has generic holonomy and P is very ample, then the generic smooth surface $\Sigma \in \mathcal{M}_P$ has $b_1(\Sigma) = 0$.

The lattice $H^2(\Sigma, \mathbb{Z})$ has an intersection form. It is embedded in the vector space $H^2(\Sigma, \mathbb{R})$, and the latter can be decomposed orthogonally into $H^{2,+} \oplus H^{2,-}$. As Σ moves in the moduli space the decomposition "rotates" relative to $H^2(\Sigma, \mathbb{Z})$. This is described as a variation of weight two Hodge structures. See [105] for a discussion in the present context. There is a fixed part, $L_X = \iota^* H^{1,1}(X, \mathbb{Z})$ which does not rotate. As explained in [2] L_X has signature $(1, h^{1,1}(X) - 1)$ by the Hodge index theorem [106] with the positive direction being the Kähler class J. Thus $H^{2,+}$ is spanned by the J and the $(2,0) + (0,2)$ forms, while $H^{2,-}$ is a negative definite space spanned by the orthogonal $(1,1)$ forms. Note in particular that since L_X has a nondegenerate form the matrix $D_{ABC}P^C$ is an invertible matrix — a fact we often use.

An important role in this paper is played by the locus $NL(F)$ defined by choosing $F \in H^2(\Sigma, \mathbb{Z})$ and considering the locus of divisors for which F is of type $(1,1)$. This is known as the *Noether-Lefschetz locus*, and, we are told, is a somewhat mysterious object mathematically. In [107] it is shown that $NL(F)$ is an algebraic variety. The moduli space $\mathcal{M}_{F,N}$ appearing in eq.(2.7) projects to $NL(F)$. The fiber over a smooth element $\Sigma \in NL(F)$ is $Hilb^N(\Sigma)$. Unfortunately, complicated things happen at the discriminant locus so this is not a practical way of understanding the $d(F, N)$.

C. Finiteness of the number of split attractor flows

Throughout the paper we have assumed the following statement:

The number of distinct split attractor flows terminating on regular attractor points, beginning with a fixed charge Γ_0, at a fixed initial point t_∞, is finite.

In this appendix we will prove a weaker version of this claim, namely that the number of attractor flows terminating in any fixed compact region of Teichmüller space is finite. In fact our argument proves rather more and addresses a class of noncompact regions. The proof uses some general ideas mentioned in appendix A of [23].

We will be using the large Kähler structure formulae for the central charges. Expansions around this point in moduli space distinguish a duality frame of electric and magnetic charges. The first step in the argument shows that there are a finite number of possible collections of magnetic charges of the final regular attractor points. To do this we consider the attractor equation for a charge Γ written as:

$$2\mathrm{Im}(\bar{Z}(\Gamma)Z(\Gamma')) = \langle \Gamma, \Gamma' \rangle \tag{C.1}$$

for all charges Γ'. It therefore follows that

$$|Z(\Gamma;t_*(\Gamma))| \geq \frac{1}{2}\frac{|\langle\Gamma,\Gamma'\rangle|}{|Z(\Gamma';t_*(\Gamma))|} \tag{C.2}$$

for all Γ' such that $Z(\Gamma';t_*(\Gamma)) \neq 0$.

Let us consider first the one-dimensional case with $\Gamma = r + bP + cP^2 + dP^3$ and $t_*(\Gamma) = (x+iy)P$. Then applying the inequality (C.2) with $\Gamma' = P^3$ we get

$$|Z(\Gamma;t_*(\Gamma))| \geq \frac{1}{2}\sqrt{\frac{4P^3}{3}}|r|y^{3/2} \tag{C.3}$$

and using $\Gamma' = P^2$ we get

$$|Z(\Gamma;t_*(\Gamma))| \geq \frac{1}{2}\sqrt{\frac{4P^3}{3}}|b|\frac{y^{3/2}}{|x+iy|} \tag{C.4}$$

In order to for these inequalities to be useful we must assume that the final regular attractor points will be contained in a region of Teichmüller space of the form $-L \leq x \leq L$, $y \geq y_m$. These are the noncompact regions alluded to above. The need to restrict attention to such regions is the main limitation of the present argument.

Granted that the flows lie in a region of the above type, we have absolute lower bounds at attractor points:

$$|Z(\Gamma;t_*(\Gamma))| \geq \frac{1}{2}\sqrt{\frac{4P^3}{3}}|r|y_m^{3/2} \tag{C.5}$$

and using $\Gamma' = P^2$ we get

$$|Z(\Gamma;t_*(\Gamma))| \geq \frac{1}{2}\sqrt{\frac{4P^3}{3}}|b|\frac{y_m^{3/2}}{|L+iy_m|} \tag{C.6}$$

Let us absorb the factor $\sqrt{P^3/3}$ into Z to define $\hat{Z}$.

Now, if we consider an attractor flow tree starting from (Γ_0, t_∞) then since the flow is gradient flow for $\log|Z(\Gamma;t)|^2$, and since at the walls of marginal stability with vertices $\Gamma \to \Gamma_1 + \Gamma_2$, we have $|Z(\Gamma;t_{ms})| = |Z(\Gamma_1;t_{ms})| + |Z(\Gamma_2;t_{ms})|$, we see that if the final regular attractor points for the flow tree are labelled (Γ_i, t_i), $i = 1, \ldots N$ then we have

$$|\hat{Z}(\Gamma_0;t_\infty)|\frac{1}{y_m^{3/2}} \geq \sum_{i=1}^{N}|r_i| \tag{C.7}$$

$$|\hat{Z}(\Gamma_0;t_\infty)|\frac{|L+iy_m|}{y_m^{3/2}} \geq \sum_{i=1}^{N}|b_i| \tag{C.8}$$

Because charges are quantized r_i are integers, and (taking P to be primitive, for simplicity) b_i are integers. It follows that there are a finite number of sets of possible final magnetic charges $\{(r_1,b_1),\ldots,(r_N,b_N)\}$. From this finite list of charges we only keep those for which $\sum r_i = r$ and $\sum b_i = b$. Then from the remaining list there are a finite number of topologies of binary trees we can build up from these final charges terminating on a single initial charge. Let us call these "magnetic flow trees."

The finiteness of the number of magnetic flow trees does not yet imply that there are a finite number of attractor flow trees because we have not taken into account the electric charges. Now, the the regular attractor flows for pure electric charges, i.e. for $D2D0$ boundstates, goes to $t = i\infty$. For this reason the inequalities we get taking Γ' to be a magnetic charge are less useful and we need to use a different kind of argument.

Suppose there were an infinite set of attractor flow trees. As we have seen there is a finite list of magnetic flow trees terminating on regular attractor points at finite places in moduli space. Therefore, there would have to be an infinite family of flow trees with all the $D2D0$ emissions taking place along one particular line segment taken from one particular magnetic flow tree. We are not allowing splits where all three charges have zero magnetic charge, and hence this line-segment must carry some nonzero magnetic charge $(r_*, b_* P) \neq 0$. Order the infinite set of trees with electric emissions from this line segement and let $(c_\alpha P^2, d_\alpha P^3)$, $\alpha = 1, \ldots, \infty$ be the electric charges emitted from this line segment in the ensemble of all trees. Let the point at which they are emitted be $t_\alpha = (x_\alpha + i y_\alpha)P$. Let the charge along the line right after this emission be $(r_*, b_* P, \hat{c}_\alpha P^2, \hat{d}_\alpha P^3)$. Finally, if a flow emits $(c_\alpha P^2, d_\alpha P^3)$ then there will be a set S_α of numbers $\beta \leq \alpha$ accounting for all the electric charges $(c_\beta P^2, d_\beta P^3)$ emitted up to that point along that segment in that particular flow.

Note that we have

$$|\hat{Z}(\Gamma_0; t_\infty)| \geq |Z(r_*, b_* P, \hat{c}_\alpha P^2, \hat{d}_\alpha P^3; t_\alpha)| + \sum_{\beta \in S_\alpha} |Z(0, 0, c_\beta P^2, d_\beta P^3; t_\beta)| \tag{C.9}$$

for all α.

Now, if the set (c_α, d_α) is not bounded in $\mathbb{Z}^2$ then we clearly must have $y_\alpha \to \infty$ so that

$$|Z(0, 0, c_\alpha P^2, d_\alpha P^3; t_\alpha)| = \frac{|c_\alpha(x_\alpha + i y_\alpha) - d_\alpha|}{y_\alpha^{3/2}} \sqrt{\frac{3P^3}{4}} \tag{C.10}$$

remains bounded. On the other hand, suppose the set of electric charges (c_α, d_α) does remain bounded (for example, suppose there is an infinite set of attractor trees where more and more D2D0 lines are emitted but the charges come with alternate signs and balance each other). Nevertheless, because the sum on β in (C.9) remains bounded it must be that there is a subsequence such that

$$\frac{|(c_\alpha x_\alpha - d_\alpha) + i c_\alpha y_\alpha|}{y_\alpha^{3/2}} \to 0 \tag{C.11}$$

This still implies that $y_\alpha \to \infty$. One might wonder if we can have the numerator tend to zero while y_α remains bounded. Clearly, because of the term $i c_\alpha y_\alpha$, the c_α must have an infinite subsequence with all but finitely many zero. But then we cannot have $(c_\alpha x_\alpha - d_\alpha) \to 0$ without $d_\alpha = 0$, but then we don't have an infinite number of nonzero D2D0 charges. Thus, we must have an infinite subsequence with $y_\alpha \to \infty$.

Now, because of (C.9) we also have an upper bound (namely, $\sqrt{\frac{4}{3P^3}}|\hat{Z}(\Gamma_0;t_\infty)|$) on

$$|\frac{\frac{1}{6}r_*(x_\alpha+iy_\alpha)^3-\frac{1}{2}b_*(x_\alpha+iy_\alpha)^2}{y_\alpha^{3/2}}+\frac{\hat{c}_\alpha(x_\alpha+iy_\alpha)-\hat{d}_\alpha}{y_\alpha^{3/2}}| \tag{C.12}$$

The ensemble of complex numbers in the absolute value sign is clearly bounded. Now consider the ensemble of complex numbers

$$\frac{\hat{c}_\alpha(x_\alpha+iy_\alpha)-\hat{d}_\alpha}{y_\alpha^{3/2}}$$

We can write:

$$\frac{\hat{c}_\alpha(x_\alpha+iy_\alpha)-\hat{d}_\alpha}{y_\alpha^{3/2}}=\frac{c_*(x_\alpha+iy_\alpha)-d_*}{y_\alpha^{3/2}}-\frac{(\sum_{\beta\in S_\alpha}c_\beta)(x_\alpha+iy_\alpha)-(\sum_{\beta\in S_\alpha}d_\beta)}{y_\alpha^{3/2}} \tag{C.13}$$

for some fixed c_*, d_*. We claim this is a bounded set of complex numbers (as we let $\alpha\to\infty$). The first term on the RHS of C.13 goes to zero. For the second term we use the fact that the y_α are increasing[62] so, for example,

$$\sum_{\beta\in S_\alpha}|c_\beta|y_\alpha^{-1/2}<\sum_{\beta\in S_\alpha}|c_\beta|y_\beta^{-1/2} \tag{C.14}$$

but the RHS of this inequality is bounded. Now, since $y_\alpha\to\infty$ the ensemble of complex numbers

$$\frac{\frac{1}{6}r_*(x_\alpha+iy_\alpha)^3-\frac{1}{2}b_*(x_\alpha+iy_\alpha)^2}{y_\alpha^{3/2}}$$

in (C.12) is unbounded. This is a contradiction with the boundedness of the set (C.12) so we conclude that there can only be a finite number of split attractor flows.

The above argument can be adapted to the general case as follows.

Let the terminal regular attractor points at finite points of moduli space $t_i=B_i+iJ_i$ have magnetic charges (r_i,P_i). Using the basic inequality (C.2) with $\Gamma'=\omega$, where ω is a unit volume form we again find

$$|Z(\Gamma_0;t_\infty)|\geq\sum_i|r_i|\sqrt{\frac{J_i^3}{3}}\geq\sqrt{\frac{J_{min}^3}{3}}\sum|r_i| \tag{C.15}$$

where as before there is a lower bound on the volume in the moduli space.

Similarly, we find

$$|Z(\Gamma_0;t_\infty)|\geq\sum_i\frac{|q_2^i\cdot P_i|}{|q_2^i\cdot t_i|}\sqrt{\frac{J_i^3}{3}} \tag{C.16}$$

Here q_2^i are arbitrary charges in $H^4(X,\mathbb{Z})$ applied to each of the terminal attractor points. The t_i are in the Kähler cone, as are the P_i (by the attractor equation), so the most effective choice is to take the q_2^i to range over an integral basis $\mathcal{B}$ of effective curves generating

[62]we might need to choose a subsequence for this to be the case

$H_2(X,\mathbb{Z})$. Once again we claim that in a region of Teichmüller space where the B fields are bounded, and the Kahler classes are bounded below, there is a universal lower bound for $\frac{1}{|q_2^i \cdot t_i|}\sqrt{\frac{J_i^3}{3}}$ as q_2^i ranges over $\mathcal{B}$. It follows that $\sum_i |P_i^A|$ is bounded above for each component A in the basis dual to $\mathcal{B}$.

As before, from (C.15)(C.16) we conclude that there is a finite set of possible final magnetic attractor charges, and hence a finite number of magnetic attractor trees one can make.

As before, if there are an infinite number of attractor flow trees then there must be some line segment in some tree that supports an infinite family of different trees emitting D2D0 charges (q_2^α, q_0^α). Once again,

$$\frac{|q_2^\alpha \cdot t_\alpha - q_0^\alpha V|}{\sqrt{J_\alpha^3}} \tag{C.17}$$

must go to zero for some subsequence of electric charges. Again we conclude that $J_\alpha^3 \to \infty$ is necessary, and now observe that there is an upper bound on

$$|\frac{\frac{1}{6}r_* t_\alpha^3 - \frac{1}{2}P_* t_\alpha^2}{\sqrt{J_\alpha^3}} + \frac{\hat{q}_2^\alpha t_\alpha - \hat{q}_0^\alpha V}{\sqrt{J_\alpha^3}}| \tag{C.18}$$

The above is a sum of two complex numbers. The ensemble formed by the second (as $\alpha \to \infty$) is bounded, but the first cannot be, but this contradicts the fact that the norm is bounded.

Thus, there must be a finite number of split attractor flows terminating in the regions of the type we have specified. For physical reasons we firmly believe that the number of split attractor flows in all of Teichmüller space is unconditionally finite. Unfortunately, it appears to us that the above ideas are not sufficiently powerful to prove this, and the proof will need a new idea.

D. Attractor tree numerics

As reviewed in section 3.2, whenever the entropy function S on charge space is known explicitly, one can explicitly construct all solutions to the BPS equations of motion. In particular one can in principle explicitly construct attractor flows and the trees built from them, although explicit expressions often become very complicated. The same explicit prescriptions can be used though to construct highly efficient numerical algorithms, e.g. for determining whether or not a tree of given topology exists in a given background.

In this appendix, we sketch such an algorithm and explain how we used it to check the extreme polar state conjecture.

D.1 Existence of flow trees

The topological data of a flow tree can be specified as a nested list. For example the tree sketched in fig. 20 is represented as $\mathcal{T} = \{\{\Gamma_1, \Gamma_2\}, \{\Gamma_3, \{\Gamma_4, \Gamma_5\}\}\}$.

To determine whether a split $\Gamma \to \Gamma_1 + \Gamma_2$ exists starting from some initial point $t_{\rm in}$, i.e. whether a wall of marginal stability is crossed before an attractor point or zero of $Z(\Gamma)$ is reached, we proceed as follows. We parametrize the attractor flow in the usual way by $\tau = 1/r$ such that the initial point corresponds to $\tau = 0$ and $U_{\tau=0} = 0$. Note that from (3.25) we always find a unique value of τ where $\mathrm{Im}(Z_1 \bar{Z}_2) = 0$:

$$\tau_0 = \frac{2}{\langle \Gamma_1, \Gamma_2 \rangle} \frac{\mathrm{Im}(Z_1 \bar{Z}_2)}{|Z|}\bigg|_{\tau=0}. \qquad \text{(D.1)}$$

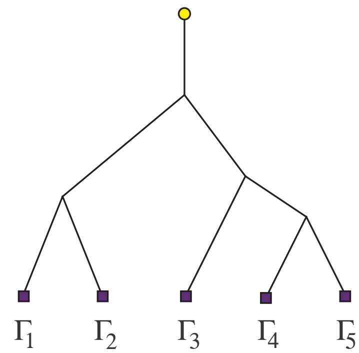

Figure 20: Example of topological flow tree data.

When the entropy function S is known explicitly, the value of the moduli t^A and therefore the central charges Z_1 and Z_2 at $\tau = \tau_0$ can be computed explicitly using the results of [24]. For example in the (effective) one modulus, large radius case with $(p^0, p, q, q_0) = p^0 + pD + qD^2 + q_0 D^3$ with $D \equiv D_1$ the basis divisor, we have

$$\begin{aligned} S(p^0, p, q, q_0) &= \frac{\pi}{3} \sqrt{3\, p^2\, q^2 - 8\, p^0\, q^3 - 6\, p^3\, q^0 + 18\, p\, p^0\, q\, q^0 - 9\, p^{0\,2}\, q^{0\,2}} \\ t(p^0, p, q, q_0) &= \frac{p\, q - 3\, p^0\, q^0 + i \sqrt{3\, p^2\, q^2 - 6\, p^3\, q^0 + 18\, p\, p^0\, q\, q^0 - 8\, p^0\, q^3 - 9\, p^{0\,2}\, q^{0\,2}}}{p^2 - 2\, p^0\, q}, \end{aligned}$$

so the flows are given by $t(\tau) = t(H(\tau))$ with $H(\tau) = -\Gamma\, \tau + 2\, \mathrm{Im}(e^{-i\alpha}\Omega)_{\tau=0}$.

The value of τ_0 given by (D.1) corresponds to an actual split point if and only if $\tau_0 > 0$, $\mathrm{Re}(Z_1 \bar{Z}_2)|_{\tau_0} > 0$, and $t^A|_{\tau_0}$ lies in the interior of Teichmüller space (in the large radius approximation in which we work this amounts to $\mathrm{Im}\, t^A|_{\tau_0} > 0$).

To determine whether the full flow tree exists, it thus suffices to check recursively through the nested list for the existence of the subsequent splits, as outlined above, and finally whether the Γ_i attractor points exist for the endpoints of the tree. In the large radius approximation, this is equivalent to having positive discriminant $S^2(\Gamma_i) \sim \mathcal{D}(\Gamma_i)$.

All of this is easily done numerically. A straightforward implementation in Mathematica manages to check about one thousand splits per second on a 2 GHz Pentium.

D.2 Maximizing $\widehat{q}_0$

Using the procedure for checking the existence of flow trees sketched above, we can try to find numerically the maximally polar states (i.e. maximal $\hat{q}_0$) within a specified ensemble of flow trees, thus providing evidence for the extreme polar state conjecture of section 6.2.2.

We implemented this in Mathematica by a simple random walk algorithm, starting from an existing attractor flow tree within an ensemble specified by a flow tree topology and charges $\Gamma_i(u)$ depending on a set of parameters u. At each step random points u near the latest successful point $u_{\rm prev}$ are chosen until a value of u is found which gives an actual attractor flow tree with $\hat{q}_0$ larger than the maximal $\hat{q}_0$ so far, with some bias in the direction of the last successful step. If the number of trials exceeds a certain cutoff, the

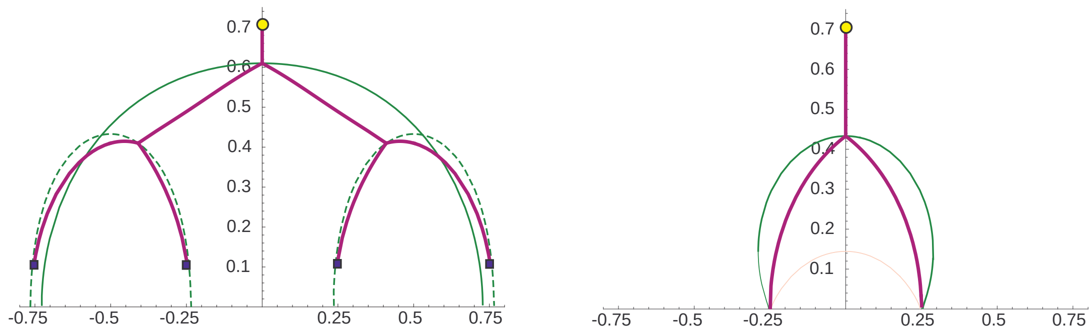

Figure 21: Left: Initial four-legged flow tree in D6-D6-anti-D6-anti-D6 optimization procedure. **Right**: Endpoint of optimization. The final two splits have become invisibly small; in spacetime this flow tree corresponds to near-coincident $\mathrm{D6}-\mathrm{D6}$ and near-coincident $\overline{\mathrm{D6}}-\overline{\mathrm{D6}}$, with nearly pure fluxed D6 and anti-D6 branes.

stepsize is decreased. This goes on till a specified (large) number of attractor flow trees has been evaluated. The whole process is repeated several times over, eliminating the less successful random walks.

For example, we considered four centered $\mathrm{D6}-\mathrm{D6}-\overline{\mathrm{D6}}-\overline{\mathrm{D6}}$ flow trees with topology as in fig. 21a and charges parametrized by $\Gamma_1 = e^{P/4+S_a}(1-\tilde{\beta}D^2+\tilde{n}_1D^3)$, $\Gamma_2 = e^{P/4-S_a}(1-\tilde{\beta}D^2+\tilde{n}_2D^3)$, $\Gamma_3 = -e^{-P/4-S_b}(1-\tilde{\beta}D^2+\tilde{n}_3D^3)$, $\Gamma_4 = -e^{P/4+S_b}(1-\tilde{\beta}D^2+\tilde{n}_4D^3)$. Keeping P fixed at $P=1$ and starting at $\{S_a, S_b, \beta, \tilde{n}_1, \tilde{n}_2, \tilde{n}_3, \tilde{n}_4\} = \{5\times 10^{-2}, 5\times 10^{-2}, 1.92\times 10^{-2}, 2.03\times 10^{-3}, -2.03\times 10^{-3}, -2.03\times 10^{-3}, 2.03\times 10^{-3}\}$ (shown in fig. 21a), running 100 times at a cutoff of $100,000$ flow tree evaluations, resulted in a maximal $\hat{q}_0$, $(\hat{q}_0)_{\max} = 0.0104064$ at $\{S_a, S_b, \beta, \tilde{n}_1, \tilde{n}_2, \tilde{n}_3, \tilde{n}_4\} = \{4.9\times 10^{-4}, 1.5\times 10^{-4}, 1.0\times 10^{-5}, 3.1\times 10^{-8}, -3.1\times 10^{-8}, -3.1\times 10^{-8}, 7.4\times 10^{-9}\}$ (shown in fig. 21b). This is fully compatible with our conjectured $(\hat{q}_0)_{\max} = P^3/24r^2 = 1/96 \approx 0.0104167$, at $\{S_a, S_b, \beta, \tilde{n}_1, \tilde{n}_2, \tilde{n}_3, \tilde{n}_4\} = \{0,0,0,0,0,0,0\}$, in accordance with the extreme polar state conjecture.

E. The three node quiver index

In this appendix we evaluate the integral (5.47) yielding the Euler characteristic of $\mathcal{M}$ defined in (5.44). We therefore define the function:

$$\chi(a,b;c) := \oint dJ_1 \oint dJ_2 J_1^{-a} J_2^{-b} \frac{(1+J_1)^a(1+J_2)^b}{(1+J_1+J_2)^c}(J_1+J_2)^c. \tag{E.1}$$

In this appendix we will derive the following four main properties.

First, we obviously have $\chi(a,b;c) = \chi(b,a;c)$. Second, we can write $\chi(a,b;c)$ in terms of an integral of Laguerre polynomials:

$$\chi(a,b;c) = ab - \int_0^\infty ds e^{-s} L^1_{a-1}(s) L^1_{b-1}(s) L^1_{c-1}(s) \tag{E.2}$$

in particular, $\chi(a,b;c) = ab - f(a,b,c)$ where $f(a,b,c)$ is totally symmetric.

To state the third and fourth properties note that for 3 positive integers a, b, c either all three triangle inequalities are satisfied

$$a + b \geq c \tag{E.3}$$

$$b + c \geq a \tag{E.4}$$

$$c + a \geq b \tag{E.5}$$

or precisely one is violated. Our third property states that:

$$\chi(a,b;c) = \begin{cases} b(a-c) \text{ if } & a \geq b+c \\ a(b-c) \text{ if } & b \geq a+c \\ 0 \quad\quad\quad \text{if } & c \geq a+b \end{cases} \tag{E.6}$$

Fourth, when all three inequalities (E.3) are satisfied we do not have a simple formula for $\chi(a,b;c)$, but we do have the asymptotic formula

$$\chi(a,b;c) \sim k(-1)^{a+b+c}(abc)^{-1/3}2^a 2^b 2^c \tag{E.7}$$

where k is a constant.

E.1 Derivation of property two

Write $\chi(a,b;c)$ as:

$$\chi(a,b;c) = \frac{1}{(c-1)!}\oint dx_1 \oint dx_2 \, (1+1/x_1)^a \, (1+1/x_2)^b \, (x_1+x_2)^c \int_0^\infty \frac{ds}{s} s^c e^{-s(1+x_1+x_2)} \tag{E.8}$$

$$= \frac{1}{(c-1)!}\int_0^\infty \frac{ds}{s} s^c e^{-s} \oint dx_1 \oint dx_2 \, (1+1/x_1)^a \, (1+1/x_2)^b \left(-\frac{\partial}{\partial s}\right)^c e^{-s(x_1+x_2)} \tag{E.9}$$

$$= \frac{1}{(c-1)!}\oint_0^\infty \frac{ds}{s} s^c e^{-s} \left(-\frac{\partial}{\partial s}\right)^c \left[\left(\oint dx_1 \, (1+1/x_1)^a \, e^{-sx_1}\right)\left(\oint dx_2 \, (1+1/x_2)^b \, e^{-sx_2}\right)\right] \tag{E.10}$$

Now we note that

$$\oint dx_1 \, (1+1/x_1)^a \, e^{-sx_1} = \sum_{j=1}^{a} \binom{a}{j} \frac{(-s)^{j-1}}{(j-1)!} = L^1_{a-1}(s) \tag{E.11}$$

is a Laguerre polynomial. Thus we can write

$$\chi(a,b;c) = \frac{1}{(c-1)!}\oint_0^\infty ds s^{c-1} e^{-s} \left(-\frac{\partial}{\partial s}\right)^c \left(L^1_{a-1}(s) L^1_{b-1}(s)\right) \tag{E.12}$$

Now integrate by parts $c-1$ times (assuming $c-1 > 0$). The boundary terms do not contribute. Next use the Rodrigues formula

$$\left(\frac{d}{dx}\right)^n (x^{n+a} e^{-x}) = n! x^a e^{-x} L^a_n(x) \tag{E.13}$$

to get

$$\chi(a,b;c) = \int_0^\infty ds e^{-s} L^0_{c-1}(s) \left(-\frac{\partial}{\partial s}\right) \left(L^1_{a-1}(s) L^1_{b-1}(s)\right) \tag{E.14}$$

The last integration by parts produces

$$\chi(a,b;c) = ab + \int_0^\infty ds \frac{d}{ds}(e^{-s} L^0_{c-1}(s)) L^1_{a-1}(s) L^1_{b-1}(s) \tag{E.15}$$

Finally, using

$$\frac{d}{ds}(e^{-s} L^0_{c-1}(s)) = -e^{-s} L^1_{c-1}(s) \tag{E.16}$$

we arrive at the elegant formula (E.2). Of course the integral can be done explicitly as a triple sum:

$$\chi(a,b;c) = ab - \sum_{s=0}^{a-1}\sum_{t=0}^{b-1}\sum_{u=0}^{c-1} \binom{a}{s+1}\binom{b}{t+1}\binom{c}{u+1} \frac{(s+t+u)!}{s!t!u!}(-1)^{s+t+u} \tag{E.17}$$

E.2 Evaluation when a triangle inequality is violated

The Laguerre form (E.2) of the function does not appear to be the most useful form for evaluating χ in this region. Rather, in the contour integral (E.1) it is useful to make the change of variables

$$z_i := 1 + 1/J_i \tag{E.18}$$

The contour will now be on two *large* circles with radius $\cong 1/\epsilon_i$ and we have the contour integral

$$\chi(a,b;c) = \mathcal{I}(a,b;c) = \oint dz_1 \oint dz_2 \frac{1}{(1-z_1)^2} \frac{1}{(1-z_2)^2} z_1^a z_2^b \left(\frac{z_1+z_2-2}{z_1 z_2 - 1}\right)^c \tag{E.19}$$

Let us try to do the integral by deforming the z_1 contour first. Then we potentially pick up poles at $z_1 = 1$ and $z_1 = 1/z_2$. This leads to $\mathcal{I} = \mathcal{I}_1 + \mathcal{I}_2$ where $\mathcal{I}_1$ comes from the pole at $z_1 = 1$ and $\mathcal{I}_2$ from the pole at $z_1 = 1/z_2$. We have

$$\mathcal{I}_1 = \oint dz_2 \frac{d}{dz_1}|_{z_1=1} \left[\frac{z_2^b}{(z_2-1)^2} z_1^a \left(\frac{z_1+z_2-2}{z_1 z_2 - 1}\right)^c\right] \tag{E.20}$$

$$\mathcal{I}_2 = \oint dz_2 \frac{1}{(c-1)!} \left(\frac{d}{dz_1}\right)^{c-1} |_{z_1=1/z_2} \left[\frac{z_2^{b-c}}{(z_2-1)^2} \frac{z_1^a}{(z_1-1)^2} (z_1+z_2-2)^c\right] \tag{E.21}$$

It is straightforward to carry out the differentiation in $\mathcal{I}_1$ and evaluate the z_2 integral from its pole at $z_2 = 1$:

$$\mathcal{I}_1 = \oint dz_2 (a-c) \frac{z_2^b}{(z_2-1)^2} = (a-c)b \tag{E.22}$$

In order to evaluate $\mathcal{I}_2$ we expand

$$(z_1+z_2-2)^c = \sum_{s=0}^{c} \binom{c}{s} (z_1-1)^s (z_2-1)^{c-s}$$

so now we write:

$$\mathcal{I}_2 = \mathcal{I}_2^A + \mathcal{I}_2^B + \mathcal{I}_2^C \tag{E.23}$$

$$
\begin{aligned}
\mathcal{I}_2^A &= \oint dz_2 \frac{1}{(c-1)!}\left(\frac{d}{dz_1}\right)^{c-1}|_{z_1=1/z_2}\left[\frac{z_2^{b-c}}{(z_2-1)^2}\frac{z_1^a}{(z_1-1)^2}(z_2-1)^c\right] \\
\mathcal{I}_2^B &= \oint dz_2 \frac{1}{(c-1)!}\left(\frac{d}{dz_1}\right)^{c-1}|_{z_1=1/z_2}\left[\frac{z_2^{b-c}}{(z_2-1)^2}\frac{z_1^a}{(z_1-1)^2}c(z_1-1)(z_2-1)^{c-1}\right] \\
\mathcal{I}_2^C &= \oint dz_2 \frac{1}{(c-1)!}\left(\frac{d}{dz_1}\right)^{c-1}|_{z_1=1/z_2}\left[\frac{z_2^{b-c}}{(z_2-1)^2}\sum_{s=2}^{c}\binom{c}{s} z_1^a(z_1-1)^{s-2}(z_2-1)^{c-s}\right]
\end{aligned}
$$

Next we write out the action of the derivative wrt z_1. Important simplifications occur because after differentiating we replace

$$z_1 - 1 \to -z_2^{-1}(z_2-1)$$

The remaining z_2 integral will get contributions from the poles at $z_2 = 1$ and, possibly, at $z_2 = 0$. If we replace

$$
\begin{aligned}
&\left(\frac{d}{dz_1}\right)^{c-1}|_{z_1=1/z_2} z_1^a (z_1-1)^{s-2} \\
&= \sum_{k=0}^{c-1}\binom{c-1}{k}\frac{a!}{(a-(c-1-k))!}\frac{(s-2)!}{(s-2-k)!} z_1^{a-(c-1-k)}(z_1-1)^{s-2-k}
\end{aligned}
$$

then we find after setting $z_1 = 1/z_2$ that the term is proportional to

$$z_2^{b-a-s+1}(z_2-1)^{c-k-4} \tag{E.24}$$

Thus, we can only get a pole for the contributions from $k = c-3, c-2, c-1$ and then we find that only $s = c, c-1$ can contribute.

Adding up the contributions we get

$$\mathcal{I}_2^C = a(a-1) + a(b-a-c+1) + \oint_{z_2=0} [...] \tag{E.25}$$

The second term arises from the contributions of the poles at $z_2 = 0$. From (E.24) we see that these poles are absent if $b+1 \geq a+c$.

In exactly the same way we find that

$$
\begin{aligned}
\mathcal{I}_2^A &= \frac{1}{2}a(a-1)(c-2) + a(c-1)(b-a+1) + \frac{1}{2}c(b-a+1)(b-a) + \oint_{z_2=0}[...] \\
\mathcal{I}_2^B &= -\frac{1}{2}a(a-1)c - ac(b-a) - \frac{1}{2}c(b-a)(b-a-1) + \oint_{z_2=0}[...]
\end{aligned}
$$

where we have added up the poles at $z_2 = 1$. The poles at zero are absent for $b+1 \geq a$ for $\mathcal{I}_2^A$ and $b \geq a$ for $\mathcal{I}_2^B$.

Thus, when $b+1 \geq a+c$ we only have poles from $z_2 = 1$ and adding up the contributions we find

$$\chi = \mathcal{I}(b,a;c) = a(b-c) \qquad\qquad a+c \leq b+1, \tag{E.26}$$

in agreement with what we found in section 5.3.

What about other ranges? Clearly we have $\mathcal{I}(b,a;c) = \mathcal{I}(a,b;c)$ so we also have

$$\chi = \mathcal{I}(a,b;c) = b(a-c) \qquad\qquad b+c \le a+1 \tag{E.27}$$

When $c \ge a+b$ we can use the fact that $\chi(a,b;c) = ab - f(a,b,c)$ with $f(a,b,c)$ totally symmetric to derive $\chi(a,b;c) = 0$.

E.3 Large (a,b,c) asymptotics

For estimating asymptotics at large a,b,c satisfying the triangle inequality we return to the formula (E.2). For simplicity we consider

$$\int_0^\infty dse^{-s} L_A^1(s) L_B^1(s) L_C^1(s)$$

For large A,B,C the integrand oscillates, but has a large peak (at least for $A=B=C$) and one can try to do the integral by saddle-point approximation. The integrand is certainly very small for $s \ge (A+B+C)$.

The appropriate asymptotic expansion of the Laguerre polynomials for our needs is that given in [112], equation 8.22.10. Namely, for $x = (4n+4)\cosh^2(\phi)$, $\epsilon \le \phi \le \Lambda$

$$L_n^1(x) \sim \frac{1}{2}(-1)^n e^{x/2} (\pi \sinh\phi)^{-1/2} x^{-3/4} n^{1/4} \exp\Big((n+1)(2\phi - \sinh 2\phi)\Big)(1 + \mathcal{O}(n^{-1})) \tag{E.28}$$

This covers a region up to $(4n+4)\cosh^2\Lambda$ for any fixed Λ as $n \to \infty$. We see that the integrand grows much more slowly than $e^{-x+3x/2}$ in this region. Beyond this region we will start to get exponential decay.

We consider the case where A,B,C do not differ too much from some common large integer N. To be more precise, define ϕ_A, ϕ_B, ϕ_C by $s = 4(A+1)\cosh^2\phi_A$, etc. and define also $s = 4(N+1)\cosh^2\phi$. We define $\mu_A := \frac{A+1}{N+1} := 1 + \delta\mu_A$ and we are considering limits where $\delta\mu_A = (A-N)/(N+1) \sim N^{-\theta}$ with $0 < \theta < 1$. We will neglect corrections to the integral of order $1 + \mathcal{O}(\delta\mu)$. These are very complicated. But we will keep corrections to the entropy of order $(N+1)\delta\mu^2 \sim N^{1-2\theta}$. Note that if $1/2 > \theta$ these are even dominant over the $\log N$ correction from the one-loop prefactor.

We solve $\cosh\phi_A = (1+\delta\mu_A)^{-1/2}\cosh\phi$ by

$$\phi_A = \phi - \frac{1}{2}\delta\mu_A \coth\phi + \frac{\delta\mu_A^2}{16}\frac{(\cosh 3\phi - 3\cosh\phi)}{(\sinh\phi)^3} + \cdots \tag{E.29}$$

and then expand the action to second order:

$$f = (N+1)\Big(2\cosh^2\phi + 3(2\phi - \sinh 2\phi) + 2\phi(\delta\mu_A + \delta\mu_B + \delta\mu_C) - \frac{1}{2}(\delta\mu_A^2 + \delta\mu_B^2 + \delta\mu_C^2)\coth\phi + \cdots\Big) \tag{E.30}$$

We find the stationary point for this action is

$$\phi_* = \log\sqrt{2} + \frac{1}{2}(\delta\mu_A + \delta\mu_B + \delta\mu_C) - \frac{3}{4}(\delta\mu_A + \delta\mu_B + \delta\mu_C)^2 + (\delta\mu_A^2 + \delta\mu_B^2 + \delta\mu_C^2) \tag{E.31}$$

and the saddle point value is

$$f(\phi_*) = (N{+}1)\Big(\log 8{+}(\delta\mu_A{+}\delta\mu_B{+}\delta\mu_C)\log 2{+}\frac{1}{2}((\delta\mu_A{+}\delta\mu_B{+}\delta\mu_C)^2{-}3(\delta\mu_A^2{+}\delta\mu_B^2{+}\delta\mu_C^2)){+}\cdots\Big) \tag{E.32}$$

From this we get:

$$I_{ABC} \sim \frac{2^{1/2}}{\pi 3^{7/2}}\frac{(-1)^{A+B+C}}{(ABC)^{1/3}}2^{A+1}2^{B+1}2^{C+1} \tag{E.33}$$

$$e^{(N+1)\frac{1}{2}((\delta\mu_A+\delta\mu_B+\delta\mu_C)^2-3(\delta\mu_A^2+\delta\mu_B^2+\delta\mu_C^2))}\Big(1+\mathcal{O}(\delta\mu_A,\delta\mu_B,\delta\mu_C)\Big) \tag{E.34}$$

Note that if A, B, C are not very different from each other, as we assumed, then it is natural to take $N = (A+B+C)/3$.

Thus, translating back to our entropy we find that in this regime,

$$\chi \sim \frac{2^{1/2}}{\pi 3^{7/2}}\left(\frac{(-1)^a 2^a}{a^{1/3}}\right)\left(\frac{(-1)^b 2^b}{b^{1/3}}\right)\left(\frac{(-1)^c 2^c}{c^{1/3}}\right)(1+\cdots) \tag{E.35}$$

where the corrections in $+\cdots$ are of order

$$\mathcal{O}((\frac{2a-b-c}{b+a+c}),(\frac{2b-a-c}{b+a+c}),(\frac{2c-b-a}{b+a+c})) \tag{E.36}$$

The leading order factorization of the answer, and especially the factors 2^a etc. call for a conceptual explanation!

F. Index vs. absolute cohomology and the entropy of 5d black holes

In [3] C. Vafa adduced an example of black hole entropy counting which appears to imply that the entropy can only be accounted for by computing the total number of BPS states without signs, rather than by an index of BPS states. In this appendix we will explain that, in fact, the entropy can be correctly accounted for using an appropriate index.

The problematic example studied in [3] involves type IIA string theory on an elliptically fibered Calabi-Yau $\pi : X \to B$ with section. The BPS states in question are those obtained from wrapping $D2$ branes on a curve $C \subset B$.

Let $\hat{C} = \pi^{-1}(C)$ be the elliptically fibered surface covering C. Then, in [3] it is argued that the relevant moduli space which one should quantize to produce BPS states is

$$\amalg_{n\geq 1}\,\mathrm{Sym}^n(\hat{C}) \tag{F.1}$$

As usual, this quantization involves a Fock space based on oscillators associated with the cohomology of $\hat{C}$. For generic elliptic fibrations one has $h^{1,0}(\hat{C}) = h^{1,0}(C)$. [63] It then follows from the adjunction formula that

$$h^{1,0}(\hat{C}) = \frac{1}{2}(C\cdot C + C\cdot K_B) + 1 \tag{F.2}$$

[63] Note that this explicitly excludes the case of a direct product $\hat{C} = C \times T^2$. Our considerations below apply equally well in the direct product case.

where the intersection products are taken within the surface B, and K_B is the canonical bundle of B. Now, using equations (B.5-B.6) together with $c_2(X) = 12\sigma\pi^*(c_1(B))\mathrm{mod}\pi^*$ where σ is the section of the elliptic fibration, (see, for example, [110], eq. 7.28), we find [3]

$$h^{2,0}(\hat{C}) = \frac{1}{2}(C \cdot C - C \cdot K_B) \tag{F.3}$$

$$h^{1,1}(\hat{C}) = C \cdot C - 9C \cdot K_B + 2. \tag{F.4}$$

Now, the key to resolving the puzzle pointed out in [3] lies in considering the $SU(2) \times SU(2)$ Lefshetz decomposition of the cohomology of the moduli space. As emphasized in [95] the existence of such a double Lefshetz decomposition follows from physical reasoning, although it is not so obvious mathematically. The existence of an $SU(2) \times SU(2)$ Lefshetz decomposition of the cohomology of $\hat{C}$ is strongly suggested by the Leray spectral sequence, and we will simply assume it exists. Some rigorous results along these lines appear in [111].

Proceeding naively, the $SU(2)_R$ raising and lowering operators are constructed using the Kähler form $\omega(C)$ of the base, while those for $SU(2)_L$ are constructed using $\omega(E) := \omega(\hat{C}) - \omega(C)$ which may be regarded as the Kähler form of a generic fiber. The decomposition into multiplets of the type $(\mathbf{j}_L, \mathbf{j}_R)$ is then

$$2h^{1,0}(\hat{C})(\frac{1}{2}, 0) \oplus (2h^{2,0} + h^{1,1} - 2)(0,0) \oplus (\frac{1}{2}, \frac{1}{2}) \tag{F.5}$$

where the last summand is the multplet $1, \omega(C), \omega(E), \omega(C) \wedge \omega(E)$.

Taking into account the symmetric products (F.1) we have

$$Z = \mathrm{Tr}(-1)^{2m_L + 2m_R} y^{2m_L} q^N = \prod_{n \geq 1} \frac{\left((1 - yq^n)(1 - y^{-1}q^n)\right)^{N_f}}{(1 - q^n)^{N_b}} \tag{F.6}$$

where

$$N_f = 2h^{1,0}(\hat{C}) - 2 \tag{F.7}$$

$$N_b = h^{1,1}(\hat{C}) + 2h^{2,0}(\hat{C}) - 2. \tag{F.8}$$

Note that if we wish to extract the Euler character of the moduli spaces then we set $y = 1$ and study the coefficients of q^n. For $y = 1$ we indeed we obtain

$$\eta^{-\chi(\hat{C})} \tag{F.9}$$

where $\chi(\hat{C}) = -12C \cdot K_B$. From the Calabi-Yau condition $\chi(\hat{C}) > 0$ so this will produce exponential degeneracies $\sim \exp[\pi\sqrt{8|C \cdot K_B|n}]$ but, as stressed in [3] the growth under uniform scaling of charges $(C, n) \to (\Lambda C, \Lambda n)$ goes as $\exp[const.\Lambda]$ in contradiction with the supergravity entropy which scales like $\exp[const.\Lambda^{3/2}]$. On the other hand, since $b^{even}(\hat{C})$ and $b^{odd}(\hat{C})$ each scale like $C \cdot C$ for large C, the absolute cohomology will grow like $\exp[const.\sqrt{C \cdot C n}] \sim \exp[const.\Lambda^{3/2}]$. This observation suggests that, at least in this example, one needs to use the absolute cohomology – the sum over all BPS states without signs – to account properly for the entropy. Unfortunately, that proposal in turn leads to many paradoxes.

There is an alternative however. To account for the entropy we should work at fixed j_L, and compute the asymptotic growth of $N_Q^{m_L}$ as explained in section 6.1.2. In order to do this properly we should incorporate the Wilson line degrees of freedom for the $D2$ wrapped on C. This leads to an extra torus factor in the moduli space, and the quantization of that torus leads to a factor $y - 2 + y^{-1} = (y^{1/2} - y^{-1/2})^2$ for each T^2. Therefore, we are interested in the asymptotics of the coefficients $D'(n, \ell)$ defined by

$$(y^{1/2} - y^{-1/2})^{2h^{1,0}(\hat{C})} \prod_{n \geq 1} \frac{((1 - yq^n)(1 - y^{-1}q^n))^{N_f}}{(1 - q^n)^{N_b}} = \sum D'(n, \ell) q^n y^\ell \tag{F.10}$$

Setting $y = e^{2\pi i z}$ and $q = e^{2\pi i \tau}$ and using the product formula for the theta function we see that the asymptotics for large n of $D'(n, \ell)$ are in turn governed by those in the Fourier expansion of

$$(y^{1/2} - y^{-1/2})^2 \eta^{-\chi(\hat{C})} \left(\frac{\vartheta_1(z, \tau)}{\eta^3} \right)^{C \cdot C + K_B \cdot C} \tag{F.11}$$

We are interested in the leading behavior for $(C, n) \to (\Lambda C, \Lambda n)$ and since $\chi(\hat{C})$ is linear in C the first two factors in (F.11) lead to a subleading correction to the entropy.

Now let us derive the asymptotics of the Fourier coefficients of

$$\eta^{-\chi(\hat{C})} \left(\frac{\vartheta_1(z, \tau)}{\eta^3} \right)^{C \cdot C + K_B \cdot C} \tag{F.12}$$

Put $C^2 + C \cdot K_B = M$ and for simplicity assume M is an even integer, and define $k = M/2$. In this case (F.12) is a weak Jacobi form of index $k = M/2$ and weight $-M - \chi/2$, where $\chi = \chi(\hat{C})$. (We choose M to be even to avoid certain inconvenient phases in the modular transformations. Similarly, strictly speaking we should take χ to be a multiple of 24, but this latter point is not too essential.)

The spectral flow identity shows that (F.12) has an expansion of the form

$$\sum_{n \geq 0, \ell \in \mathbb{Z}} c(2Mn - \ell^2) q^n y^\ell \tag{F.13}$$

Decompose the sum by writing

$$\ell = \mu + 2ks \tag{F.14}$$

$$n = n_0 + ks^2 + \mu s \tag{F.15}$$

and choose a fundamental domain $-k + 1 \leq \mu \leq k$ so that we can write

$$\eta^{-\chi} (\frac{\vartheta_1(z, \tau)}{\eta^3})^M = \sum_{\mu = -k+1}^{k} H_\mu(\tau) \Theta_{\mu, k}(z, \tau) \tag{F.16}$$

where

$$H_\mu(\tau) = \sum_{n\in\mathbb{Z}} c(4kn-\mu^2) q^{n-\frac{\mu^2}{4k}-\chi/24} \tag{F.17}$$

$$= (-1)^\mu \binom{2k}{k-\mu} q^{-\frac{\mu^2}{4k}-\frac{\chi}{24}} + \cdots \tag{F.18}$$

Note that the most negative power goes like $q^{-\frac{k}{4}-\frac{\chi}{24}}$ for $\mu = \pm k$. Also note that by the modular transformations of level k theta functions the H_μ transform under $\tau \to -1/\tau$ by a finite fourier transform, times the usual modular weight of $-M - \frac{\chi+1}{2}$.

Applying the Rademacher expansion we find that for $2Mn - \nu^2 \gg 1$

$$c(2Mn-\nu^2) \sim \zeta e^{\pi\sqrt{\left(2Mn-\nu^2-\frac{M\chi}{12}\right)\left(1+\frac{\chi}{3M}\right)}} \tag{F.19}$$

with a rather awkward prefactor

$$\zeta = (-1)^{\nu+M/2}\sqrt{2}\left(\frac{M}{2}\right)^{M+\frac{1}{2}\chi+\frac{3}{2}}\left(1+\frac{\chi}{3M}\right)^{\frac{1}{2}(M+\chi/2+1)}\left(2Mn-\nu^2-\frac{M\chi}{12}\right)^{-\frac{1}{2}(M+\chi/2+2)} \tag{F.20}$$

Now, we can take into account the prefactor $y-2+y^{-1}$ in (F.11) by noting that this amounts to taking a discrete second derivative with respect to ν of (F.19). This will leave the exponential factor and modify the prefactor ζ.

Letting $M = C^2 + C\cdot K_B$ this shows that the entropy at fixed m_L is exactly that predicted macroscopically by supergravity, at least for large $2Mn-\nu^2$, and we do indeed have $\exp[const.\Lambda^{3/2}]$ growth for an index. Note that the terms depending on χ correct the leading supergravity result in an interesting way.

What has happened here is that the sum over m_L, which corresponds to putting $y=1$ leads to impressive cancellations. Nevertheless, one can still capture the entropy with an *index* rather than the absolute cohomology.

G. A derivation of $g_{\rm top}\to\infty$ OSV using flux vacua counting techniques

Now let us return to the discussion of section 2.1. As we explained below eq. (2.5), counting BPS states involves the counting of "open string flux vacua."

To make this more precise, we make use of the $\mathcal{N}=1$ special geometry structure of the D4-brane moduli space $\mathcal{M}$ [108]. Let Σ_F be the Poincaré dual 2-cycle to F, and expand Σ_F in a basis $\{C_\alpha\}$ of $H_2(P)$: $\Sigma_F = m^\alpha C_\alpha$. In a neighborhood of the divisor moduli space, parametrized by moduli z^i, $i=1,\ldots,n := h^{2,0}$, we can define chain periods Π_α by

$$\Pi_\alpha(z) := \int_{\Gamma_\alpha(z)} \Omega \tag{G.1}$$

where Γ_α is a 3-chain with a z-dependent boundary component on P given by C_α, and possibly other, fixed boundary components, independent of z. With these chain periods, we define a superpotential[64]

$$W(z) := m^\alpha \Pi_\alpha(z) = \int_{\Gamma(z)} \Omega, \tag{G.2}$$

where $\Gamma := m^\alpha \Gamma_\alpha$ is thus a 3-chain with boundary Σ_F on P. Critical points of W precisely correspond to points where $F^{(0,2)} = 0$. To see this, note that an infinitesimal holomorphic variation of W gives

$$\delta W = \int_{\delta\Gamma} \Omega = \int_{\Sigma_F} \delta n \cdot \Omega = \int_P F \wedge (\delta n \cdot \Omega)$$

where δn is the normal holomorphic vector field corresponding to the variation δz of the divisor moduli and $\delta n \cdot \Omega$ is the contraction of δn with Ω, providing an isomorphism between the space of holomorphic sections of the normal bundle to P and $(2,0)$-forms on P. Thus we see that requiring $\partial_i W = 0$ is equivalent to $F^{(0,2)} = 0$ (and therefore of course also $F^{(2,0)} = 0$).

For the same reason, we have that for each $i = 1, \ldots, h^{2,0}$, $\partial_i \Pi_\alpha(z)$ is the period vector of a $(2,0)$-form ω_i on P. The natural Kähler metric on moduli space is given by

$$g_{i\bar{j}} := \int_P \omega_i \wedge \bar{\omega}_{\bar{j}} = \partial_i \Pi_\alpha \, Q^{\alpha\beta} \, \bar{\partial}_{\bar{j}} \bar{\Pi}_\beta = \partial_i \bar{\partial}_{\bar{j}} (\Pi_\alpha \, Q^{\alpha\beta} \, \bar{\Pi}_\beta), \tag{G.3}$$

where $Q^{\alpha\beta}$ is the inverse of the intersection form $Q_{\alpha\beta} := C_\alpha \cdot C_\beta$.

Similar to the more familiar $\mathcal{N} = 2$ special geometry, acting with further derivatives on $\partial_i \Pi$ will produce periods of $(1,1)$- and $(0,2)$-forms on P, because of Griffiths transversality [108]. In particular

$$\nabla_i \partial_j \Pi(z) \sim (1,1) \tag{G.4}$$

where ∇_i is the Levi-Civita covariant derivative with respect to the above defined metric.

Let us now compute the actual BPS partition sum. For a given flux F, the number of isolated critical points of the corresponding flux superpotential W_F is given by

$$\int_{\mathcal{M}} d^{2n} z \, \delta^{2n}(\partial W_F) \, |\det \nabla_i \partial_j W_F|^2. \tag{G.5}$$

The determinant ensures that each isolated zero of the delta function contributes $+1$ to the integral. We are free to use covariant derivatives instead of ordinary derivatives because the difference is proportional to ∂W, which vanishes. At any such critical point, the divisor is frozen, so the only remaining moduli are the positions of the N D0-branes bound to P. The contribution to the total degeneracy or Euler characteristic from this component of moduli space is therefore simply $\chi(\mathrm{Sym}^N(P)) = p_\chi(N)$, where $p_\chi(N)$ are the partitions of N into χ colors.[65]

[64] This superpotential and generalizations thereof have been discussed in [43, 109].

[65] Because $b_1(X) = 0$, we have $b_1(P) = 0$ hence the Euler characteristic of the symmetric product equals the total degeneracy.

Thus, we get for the OSV black hole partition sum

$$\mathcal{Z}_{\rm BH} := \sum_q \Omega(p,q)\, e^{2\pi\phi^0 q_0 + 2\pi\phi^A q_A} \tag{G.6}$$

$$\approx \sum_{N,F} p_\chi(N)\, e^{2\pi\phi^0(-N+\frac{1}{2}F^2+\frac{\chi}{24})+2\pi\Phi\cdot F} \int_{\mathcal{M}} d^{2n}z\, \delta^{2n}(\partial W_F)\,|\det \nabla_i\partial_j W_F|^2 \tag{G.7}$$

Here $\Phi = \phi^A D_A$ viewed as an element of $H^2(P)$, and we used (2.1)-(2.4). Actually, the above partition sum misses the contributions from components which have flat directions in the divisor moduli space (since then $\det W'' = 0$), e.g. for $F = 0$. However, we will eventually make a continuous F approximation anyway, which as we will discuss is equivalent to a large q_0 or small ϕ^0 approximation, and for generic divisors the set of such components with flat directions has measure zero in flux space. So we will take the above expression for $\mathcal{Z}_{\rm BH}$ as our starting point.

The sum over N is easily performed and yields a factor $1/\eta^\chi$. We furthermore expand as before $F = m^\alpha C_\alpha$, which gives:

$$\mathcal{Z}_{\rm BH} \approx \frac{1}{\eta^\chi(e^{-2\pi\phi^0})} \int_{\mathcal{M}} d^{2n}z \sum_m e^{\pi\phi^0 Q_{\alpha\beta}m^\alpha m^\beta + 2\pi\Phi_\alpha m^\alpha}\, \delta^{2n}(m^\alpha\partial_i\Pi_\alpha)\,|\det m^\alpha\nabla_i\partial_j\Pi_\alpha|^2. \tag{G.8}$$

Note that $Q_{\alpha\beta}$ is an indefinite form of signature (b_2^+, b_2^-). However, only critical points of W contribute, at which F is in $H^{1,1}(P)$. Restricted to this space, Q has signature $(1, b_2^-)$. The one positive direction corresponds to the Kähler form J on P. This positive direction will cause the black hole partition sum to diverge, but as discussed in [12] and at length in this paper, this divergence is easily regularized by adding a Boltzmann factor $e^{-\beta H(p,q)}$. To avoid cluttering of formulas, we will not do this regularization explicitly in what follows, and use its existence only to justify formal manipulations.

Both the delta-function and the determinant can be rewritten as integrals of exponentials linear in m^α:

$$\delta^{2n}(m^\alpha\partial_i\Pi_\alpha) = \int d^{2n}\lambda\, e^{i\pi m^\alpha(\lambda^i\partial_i\Pi_\alpha + \bar\lambda^{\bar i}\bar\partial_{\bar i}\bar\Pi_\alpha)} \tag{G.9}$$

$$|\det m^\alpha\nabla_i\partial_j\Pi_\alpha|^2 = \frac{1}{\pi^{2n}}\int d^n\theta\, d^n\psi\, d^n\bar\theta\, d^n\bar\psi\, e^{\pi m^\alpha(\nabla_i\partial_j\Pi_\alpha\,\theta^i\psi^j + \bar\nabla_{\bar i}\bar\partial_{\bar j}\bar\Pi_\alpha\,\bar\theta^{\bar i}\bar\psi^{\bar j})}. \tag{G.10}$$

The second integral is over fermionic variables. This recasts the partition function (G.8) as a Gaussian ensemble with boson-fermion-fermion cubic interactions. To obtain the "large flux" asymptotics, i.e. the limit of small ϕ^0, we replace the sum over discrete fluxes m^α by an integral, parallel to [30, 31]. The resulting integral is Gaussian, so it can be performed exactly. This yields for the part of (G.8) starting at $\sum_m \approx \int d^{b_2}m$:

$$\frac{1}{\pi^{2n}}(\phi^0)^{-b_2/2} e^{-\frac{\pi}{4\phi^0}\,(2\Phi_\alpha + i\lambda^i\partial_i\Pi_\alpha + \nabla_i\partial_j\Pi_\alpha\psi^i\theta^j + \text{c.c.})\,Q^{\alpha\beta}\,(2\Phi_\beta + i\lambda^i\partial_i\Pi_\beta + \nabla_i\partial_j\Pi_\beta\psi^i\theta^j + \text{c.c.})} \tag{G.11}$$

where $b_2 := b_2(P)$ and +c.c. stands for the conjugate terms in (G.9)-(G.10). Crucial here is that $\det Q_{\alpha\beta} = 1$, because the middle cohomology of a compact manifold is always self-dual and therefore its intersection form unimodular. In the above expression and the remainder of this appendix, we drop overall phase factors.

We now need to work out the intersection products. At first sight, this seems to give a lot of complicated terms. However, the underlying $\mathcal{N}=1$ special geometry structure, and in particular Griffiths transversality, simplifies this a lot, again in parallel to the closed string case analyzed in [30, 31]. First recall that $\partial_i \Pi_\alpha \sim (2,0)$, $\nabla_i \partial_j \Pi_\alpha \sim (1,1)$, and $\Phi_\alpha \sim (1,1)$. Only intersection products of $(1,1)$ with $(1,1)$ or $(2,0)$ with $(0,2)$ can be nonzero. Furthermore, the intersection product $\Phi_\alpha\, Q^{\alpha\beta}\, \nabla_i \partial_j \Pi_\beta = 0$ because $\Phi_\alpha\, Q^{\alpha\beta}\, \partial_j \Pi_\beta = 0$ identically for all values of the moduli z.

The remaining nontrivial products can be computed using the Leibniz rule and orthogonality, together with (G.3):

$$\begin{aligned} \partial_i \Pi_\alpha\, Q^{\alpha\beta}\, \bar{\partial}_{\bar{j}} \bar{\Pi}_\beta \;&=\; g_{i\bar{j}} && \text{(G.12)} \\ \nabla_i \partial_j \Pi_\alpha\, Q^{\alpha\beta}\, \bar{\nabla}_{\bar{k}} \bar{\partial}_{\bar{l}} \bar{\Pi}_\beta \;&=\; R_{i\bar{k}j\bar{l}} && \text{(G.13)} \\ \nabla_i \partial_j \Pi_\alpha\, Q^{\alpha\beta}\, \nabla_k \partial_l \Pi_\beta \;&=:\; \mathcal{F}_{ijkl} \quad (\text{symm. in } ijkl) && \text{(G.14)} \end{aligned}$$

These are similar to (but somewhat simpler than) the closed string expressions of [30, 31].

The exponential in (G.11) thus becomes

$$e^{-\frac{\pi}{\phi^0}(\Phi^2 - \frac{1}{2} g_{i\bar{j}} \lambda^i \bar{\lambda}^{\bar{j}} + \frac{1}{2} R_{i\bar{k}j\bar{l}} \psi^i \bar{\psi}^k \theta^j \bar{\theta}^l)}. \tag{G.15}$$

The term $\mathcal{F}_{ijkl} \psi^i \theta^j \psi^k \theta^l$ drops out because $\mathcal{F}_{ijkl}$ is symmetric in its indices. Doing the Gaussian integrals over λ and $\psi, \bar{\psi}$ turns this in

$$\pi^n\, e^{-\frac{\pi}{\phi^0}\Phi^2} (\det g_{i\bar{j}})^{-1} \det(R_{i\bar{k}j\bar{l}} \theta^j \bar{\theta}^l) \tag{G.16}$$

which is equal to

$$\pi^n\, e^{-\frac{\pi}{\phi^0}\Phi^2} \det(R^k_{ij\bar{l}} \theta^j \bar{\theta}^l). \tag{G.17}$$

This can be combined with the measure $d^{2n}z$ in (G.8) to produce

$$\pi^n\, e^{-\frac{\pi}{\phi^0}\Phi^2} \det R \tag{G.18}$$

where R is the curvature 2-form

$$R^k_i = \frac{i}{2} R^k_{ij\bar{l}} dz^j \wedge d\bar{z}^{\bar{l}}. \tag{G.19}$$

We are almost ready to write down our final result. A final step is to do a modular transformation on the $1/\eta^\chi$ factor in (G.8):

$$\frac{1}{\eta^\chi(e^{-2\pi\phi^0})} = (\phi^0)^{\chi/2} \frac{1}{\eta^\chi(e^{-\frac{2\pi}{\phi^0}})}. \tag{G.20}$$

Putting everything together, and noting that $\chi = b_2 + 2$, we get (in the continuous flux / small ϕ^0 approximation):

$$\begin{aligned} \mathcal{Z}_{\rm BH} &\approx \phi^0 \frac{e^{-\frac{\pi}{\phi^0}\Phi^2}}{\eta^\chi(e^{-\frac{2\pi}{\phi^0}})} \int_{\mathcal{M}} \frac{1}{\pi^n} \det R && \text{(G.21)} \\ &\approx \hat{\chi}(\mathcal{M})\, \phi^0\, e^{\frac{2\pi}{\phi^0}(\frac{P^3 + c_2 \cdot P}{24} - \frac{\Phi^2}{2})} && \text{(G.22)} \end{aligned}$$

where

$$\hat{\chi}(\mathcal{M}) := \int_{\mathcal{M}} \frac{1}{\pi^n} \det R. \tag{G.23}$$

Alternatively

$$\Omega(p,q) \approx \hat{\chi}(\mathcal{M}) \int d\phi \, \phi^0 \, e^{-2\pi\phi\cdot q} \, e^{\frac{2\pi}{\phi^0}\left(\frac{P^3+c_2\cdot P}{24} - \frac{\Phi^2}{2}\right)}. \tag{G.24}$$

To get to (G.22) we used the small ϕ^0 approximation to the η-function and $\chi = P^3 + c_2(X)\cdot P$ (the terms dropped are exponentially suppressed). Formally (G.23) is exactly the Euler characteristic of the divisor moduli space $\mathcal{M}$, but there might be some subtleties since the metric on $\mathcal{M}$ has singularities. Note that although this is a natural result for counting critical points of W on $\mathcal{M}$, it is not trivial: while it is true that the Euler characteristic counts the number of zeros of a section of the cotangent bundle, $\partial_i W$ does not give such a section because W is not single valued on $\mathcal{M}$ (due to monodromies acting on the fluxes). Indeed, for some fluxes there will be no critical points at all, for example fluxes Poincaré dual to 2-cycles which are trivial on X, and which moreover satisfy $F^2 > 0$, cannot satisfy $F^{(0,2)} = 0$ anywhere in moduli space. Again all this has a close analog for IIB closed string flux vacua, where the analogous index is $\int \frac{1}{\pi^n} \det(R + \omega\mathbf{1})$ [30, 31]. The difference comes from the fact that the relevant covariant derivatives in the closed string case involve an additional ∂K connection piece, whose curvature is the Kähler form ω.

The moduli space for very ample divisors P is simply $\mathcal{M} = \mathbb{CP}^{I_P - 1}$, with $I_P := \frac{P^3}{6} + \frac{c_2\cdot P}{12}$. If the "differential Euler characteristic" (G.23) equals the topological Euler characteristic, we thus have

$$\hat{\chi}(\mathcal{M}) = \chi(\mathbb{CP}^{I_P-1}) = I_P. \tag{G.25}$$

The results obtained in the bulk of this paper support this assumption. (It might be possible to prove that $\hat{\chi}(\mathcal{M}) = I_P$ directly using the estimates in [119]. We have not attempted to do so.)

The result obtained here is in agreement with (2.54). The sum over S is absent here; including it is equivalent to extending the integration contour for Φ to the entire imaginary axis in (G.24). However, since the saddle point of (G.24) lies at

$$\phi^0_* = \sqrt{-\frac{P^3 + c_2 P}{24\,\hat{q}_0}} \tag{G.26}$$

$$\phi^A_* = -\phi^0 D^{AB} q_B, \tag{G.27}$$

we see that in the large q_0 limit at fixed q_A and p^A, ϕ^0_* and ϕ^A_* become small, and therefore the contributions from the extension of the integration contour or equivalently the S-shifted terms in (2.54) are actually exponentially suppressed in the regime of interest here. Hence they can be dropped consistent with our approximations.

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